PERGAMON

International Journal of Solids and Structures 37 (2000) 3965–4004

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Rheological–dynamical analogy: prediction of buckling curves of columns

Dragan D. Milašinović*

University of Novi Sad, Department of Civil Engineering, Subotica, Yugoslavia Kozaračka 2a, 24000 Subotica, Yugoslavia

Received 15 December 1998; in revised form 22 June 1999

Abstract

This paper is concerned with a new proposal regarding the analysis of inelastic deforming of materials and structures and is based on mathematical–physical analogy between rheological model and dynamical model with viscous damping. Due to the analogy, it becomes obvious that inelastic response of engineering structures is essentially a dynamical problem. The analogy exists for this specific model, and is one out of many examples of the analogies that can be observed in mechanics, as well as between mechanical and electrical (thermal, magnetic, etc.) systems, by virtue of their mathematical descriptions. Generally speaking, the rheological–dynamical analogy (RDA) is derived in order to solve the dynamical problems, but can also be used to explain any statical problems, considering the correspondent limit values of presented mathematical expressions. The paper deals with the buckling problem regarding steel, timber and concrete columns. Using the examples of typical columns, it has been demonstrated that results obtained by RDA are in good accordance with those available in cited references. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Inelastic deforming; Rheological model; Rheological–dynamical analogy; Rheological–dynamical modulus; Buckling strength; Buckling curves; Steel; Timber; Concrete

1. Introduction

Taking into consideration both the economy as well as the safety criteria as equally important, some controlled and economically justified damages of structures have to be allowed in the case of very strong time-varying loads. Balancing of the two specified conditions, as an essential goal in design engineering, consequently created the most serious problem, i.e. the necessity of understanding and predicting

* Fax: +381-24-554-580.
E-mail address: ddmil@gf.su.ac.yu (D.D. Milašinović).

PII: S0020-7683(99)00211-5

Nomenclature

A	cross sectional area
a_0,a_1,a_2	material constants
A,B,C	constant of integration
α	phase angle
B	symbol for Bingham's body: B = H–(StV|N)
Bu	symbol for Burgers's body: Bu = M–K = (N–H)–(H|N)
β_c	factor depends on initial eccentricity of the column. For solid timber, when initial eccentricity is limited on 1/300, may be taken as 0.2.
c	damping of the viscous type
C_1, C_2	constant of integration
δ	frequency ratio
E_H	slope of the elastic strain or Young's modulus
E_I	loss modulus
E_K	slope of the viscoelastic strain or viscoelastic modulus
E_K^D	dynamic viscoelastic modulus
E_R	dynamic modulus or RDA modulus
E_T	tangent modulus
E^*	complex modulus
$E_{0.05}$	modulus of elasticity, 5-percentile value
ε	total inelastic strain
$\dot{\varepsilon}$	strain rate
ε_A	cyclic strain amplitude
ε^c	creep strain
ε_{el}	elastic strain
ε_h	complementary solution of strain
ε_p	particular solution of strain
ε_p''	strain vary periodically or sinusoidal with time
ε_{ve}	viscoelastic strain
ε_{vp}	viscoplastic strain
ε''	periodic strain
$f_{c,0,d}$	design value of compression strength
$f_{c,0,k}$	characteristic value of compression strength II to the grain
f_{cm}	compression strength
φ	creep coefficient
g	gravity acceleration
γ	specific gravity
H	symbol for the Hooken spring
H'	slope of the viscoplastic strain or viscoplastic modulus
H'^D	dynamic viscoplastic modulus
I_I	loss compliance
I_R	dynamic compliance
I_z	moment of inertia (of cross-section)
I^*	complex compliance
J	symbol for the Jeffrey's body: J = N|M = N|(N–H)

K	symbol for the Kelvin's body: K = H|N
k	stiffness
k_z	radius of gyration
k_{mod}	modification factor taking into account the effect of the duration of the load and the moisture content in the structure
L	length of the column
L	symbol for the Lethersich's body: L = K–N = (H|N)–N
λ	slenderness ratio
λ_{I}	loss coefficient of normal viscosity
λ_{K}	viscoelastic coefficient of normal viscosity (Trouton's coefficient)
λ_{N}	viscoplastic coefficient of normal viscosity (Trouton's coefficient)
λ_{R}	dynamic coefficient of normal viscosity
λ^*	complex coefficient of normal viscosity
λ_{rel}	relative slenderness ratio
m	mass
M	symbol for the Maxwell's body: M = N–H
N	symbol for the Newtonian dashpot
ω	natural frequency
ω_σ	load or stress frequency
PTh	symbol for the Poynting and Thomson's body: PTh = H|M = H|(N–H)
ρ	mass density
Sch ScB	symbol for the Schofield and Scott Blair's body: Sch ScB = Schw–K = H–(StV|M)–K = H–(StV|(N–H))–(H|N)
Schw	symbol for the Schwedoff's body: Schw = H–(StV|M) = H–(StV|(N–H))
StV	symbol for Saint-Venant's body
σ	time-dependent variation of stress
$\dot{\sigma}$	stress rate
σ_{A}	cyclic stress amplitude
σ_{cr}	critical stress
σ_{p}	proportional stress
σ_{SV}	stress in the Saint-Venant's element
σ_{Y}	uniaxial yield stress
σ_0	constant stress
σ''	stress varies periodically (sinusoidal) with time
T_{K}	time of retardation
$T_{\mathrm{K}}^{\mathrm{D}}$	dynamic time of retardation
TR	symbol for Trouton and Rankine's body: TR = N–PTh = N–(H|M) = N–H|(N–H)
Y	stress level for viscoplastic yielding
|	symbol for parallel connection
–	symbol for serial connection

structural nonlinear behavior under dynamically varying loads. To model analytically the very complex material and structural inelastic behavior under dynamic loads has been a great challenge to researchers during the recent years and this remained so at the present time, too. When dealing with the development of the mathematical model for nonlinear behavior of any material, structural component or structural system, corresponding experimental tests and test results appear indispensable, because of the many involved complex phenomena directly affecting its actual inelastic response. Most of the experimental studies during the past years have been essentially conducted with such an objective.

The results of experimental research of strains show that their development and magnitude are dependent on time, and because of that rheological analysis proves unavoidable. Elasticity, plasticity, viscosity and strength are essential rheological properties from which most of the other complex properties may be derived (Reiner, 1955). The ideal bodies, typifying the three fundamental properties of elasticity, plasticity and viscosity, can be conveniently represented by the following rheological models (Hookean spring, Saint-Venant's resistance and Newton's dashpot). Combining the rheological elements, various structural rheological models illustrating the stress–strain relationships of technical media can be obtained. In the course of my research work I developed a new model of viscoelastoplastic material, that is able to describe the mutual interaction of elasticity, viscoelasticity and viscoplasticity. Based on this model, the rheological–dynamical analogy (RDA) is established as the theoretical concept for studying the inelastic material deforming. This analytical concept has been already used and proved through several inelastic problems in concrete (Milašinović, 1996, 1997).

The objective of this paper is to develop an analytical model and analytical procedure, based on new model and RDA, which can be used to predict the buckling strength and determine the buckling curves of columns. For the buckling analysis of steel columns, RDA results will be compared with the data of Smith and Sidebottom (1965). The buckling curves obtained here for wood columns will be compared with relevant buckling curves given in EC5 and in papers given by Blass (1986, 1987, 1995, 1991,). At the end the data of Hirst and Neville (1977) will be used in the example of concrete column. The aim of the paper is also to check and to prove the RDA as universal procedure for solving the various inelastic problems.

There is also a short theoretical overview of new rheological model, basic equations, which describe the RDA analytical procedure and philosophy background for determining the buckling curves using RDA in this paper. The comparison of the RDA and Euler's buckling curves is given in the form of the numerical examples, too.

2. A new rheological model

As already pointed out, the author of this paper has derived a new rheological model of a basic body which includes elastic, viscoelastic and viscoplastic strains. Every strain is in principle a function of time because a stress is always introduced into the body during a definite time interval (even a very small one) and therefore the distinction between a time-dependent strain and a strain not depending on time is very uncertain.

Here, it is assumed that the strain is measured when the specified stress has been reached. Strain ε_{el} obtained in this way, shall be considered to be independent of time, i.e. "instantaneous". Then the time-dependent, or "delayed", ε_{ve} and ε_{vp} strains are measured from the time, when the instantaneous strain has developed. Elastic material behavior can be modeled by a linear spring, E_H. Therefore, instantaneous or initial strain should be $\varepsilon_{el} = \sigma_0/E_H$ where E_H is the elastic modulus. Delayed elastic or viscoelastic strain ε_{ve} may be imagined as a common behavior of elastic E_K and viscous λ_K materials. A piston exerting pressure on a liquid with a viscosity λ_K represents ideal viscous material. Viscoelastic material behavior can be modeled by Kelvin's model, where the elastic and viscous elements are linked in parallel. The concept of delayed plastic or viscoplastic material behavior ε_{vp} may be imagined as a common behavior of the friction slider component σ_{SV} and viscous component λ_N of materials. The friction slider develops a stress σ_{SV}, becoming active only if $\sigma \geq Y$, where σ is the total applied stress and Y is some limiting yield value. The stress level in the friction slider depends on whether or not the threshold or yield stress Y, has been reached. If the stress σ is discontinued, the friction slider does not return into its original position. Viscoplastic material behavior can be modeled by the third of the sequentially linked models as shown in Fig. 1. Initial strain rate should be

$$\dot{\varepsilon} = \dot{\varepsilon}_{\mathrm{ve}} + \dot{\varepsilon}_{\mathrm{vp}} = \sigma_0/\lambda_{\mathrm{K}} + (\sigma_0 - \sigma_{\mathrm{Y}})/\lambda_{\mathrm{N}}.$$

Summarizing the above mentioned assumptions, a new rheological model may be presented by structural equation:

$$\text{H–K–(N|StV)} = \text{H–(H|N)–(N|StV)}. \tag{1}$$

The yield condition is based on the assumption of viscoplasticity

$$\sigma_{\mathrm{SV}} = \sigma(t) \;\Leftrightarrow\; \sigma(t) < Y,$$

$$\sigma_{\mathrm{SV}} = Y \;\Leftrightarrow\; \sigma(t) \geq Y, \tag{2}$$

where

$$Y = \sigma_{\mathrm{Y}} + H'\varepsilon_{\mathrm{vp}}(t). \tag{3}$$

Y is the stress level for viscoplastic yielding at any stage, σ_{Y} is uniaxial yield stress and H' is the slope of the viscoplastic strain. As stated earlier, the majority of materials are in the state of viscoelasticity in the stage of low loading, whereas after reaching the yield stress, it transits to the state of viscoplasticity. The model suggested here by explains in a simple way the transition from elastic to viscoelastic, and later to viscoplastic material, Fig. 1.

Due to the parallel connections in the model, the stresses are:

$$\sigma(t) = E_{\mathrm{H}}\varepsilon_{\mathrm{e}}(t),$$

$$\sigma(t) = E_{\mathrm{K}}\varepsilon_{\mathrm{ve}}(t) + \lambda_{\mathrm{K}}\dot{\varepsilon}_{\mathrm{ve}}(t),$$

$$\sigma(t) = H'\varepsilon_{\mathrm{vp}}(t) + \lambda_{\mathrm{N}}\dot{\varepsilon}_{\mathrm{vp}}(t) + \sigma_{\mathrm{Y}}, \tag{4}$$

while, due to the serial connections, total strain is

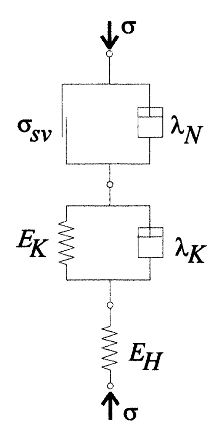


Fig. 1. A new rheological model of viscoelastoplastic body.

$$\varepsilon(t) = \varepsilon_{\mathrm{e}}(t) + \varepsilon_{\mathrm{ve}}(t) + \varepsilon_{\mathrm{vp}}(t). \tag{5}$$

Let us separate the strains and their rates from the stresses:

$$\dot{\varepsilon}_{\mathrm{e}}(t) = \frac{\dot{\sigma}(t)}{E_{\mathrm{H}}},$$

$$\dot{\varepsilon}_{\mathrm{ve}}(t) + \varepsilon_{\mathrm{ve}}(t)\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}} = \frac{\sigma(t)}{\lambda_{\mathrm{K}}},$$

$$\dot{\varepsilon}_{\mathrm{vp}}(t) + \varepsilon_{\mathrm{vp}}(t)\frac{H'}{\lambda_{\mathrm{N}}} = \frac{\sigma(t)}{\lambda_{\mathrm{N}}} - \frac{\sigma_{\mathrm{Y}}}{\lambda_{\mathrm{N}}}, \tag{6}$$

and perform the differentiation:

$$\ddot{\varepsilon}_{\mathrm{e}}(t) = \frac{\ddot{\sigma}(t)}{E_{\mathrm{H}}},$$

$$\ddot{\varepsilon}_{\mathrm{ve}}(t) + \ddot{\varepsilon}_{\mathrm{ve}}(t)\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}} = \frac{\dot{\sigma}(t)}{\lambda_{\mathrm{K}}},$$

$$\ddot{\varepsilon}_{\mathrm{vp}}(t) + \dot{\varepsilon}_{\mathrm{vp}}(t)\frac{H'}{\lambda_{\mathrm{N}}} = \frac{\dot{\sigma}(t)}{\lambda_{\mathrm{N}}}. \tag{7}$$

Summing the third part of Eq. (6), previously multiplied by $E_{\mathrm{K}}/\lambda_{\mathrm{K}}$, with the second part of Eq. (7) gives

$$\ddot{\varepsilon}_{\mathrm{ve}}(t) + \dot{\varepsilon}_{\mathrm{ve}}(t)\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}} + \dot{\varepsilon}_{\mathrm{vp}}(t)\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}} + \varepsilon_{\mathrm{vp}}(t)\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}} = \frac{\dot{\sigma}(t)}{\lambda_{\mathrm{K}}} + \frac{E_{\mathrm{K}}(\sigma(t) - \sigma_{\mathrm{Y}})}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}. \tag{8}$$

Summing the part second of Eq. (6), previously multiplied by H'/λ_{N}, with the third part of Eq. (7) gives

$$\ddot{\varepsilon}_{\mathrm{vp}}(t) + \dot{\varepsilon}_{\mathrm{vp}}(t)\frac{H'}{\lambda_{\mathrm{N}}} + \dot{\varepsilon}_{\mathrm{ve}}(t)\frac{H'}{\lambda_{\mathrm{N}}} + \varepsilon_{\mathrm{ve}}(t)\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}} = \frac{\dot{\sigma}(t)}{\lambda_{\mathrm{N}}} + \frac{H'\sigma(t)}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}. \tag{9}$$

Summing Eqs. (8) and (9) gives

$$\begin{aligned}&\left[\ddot{\varepsilon}_{\mathrm{ve}}(t) + \ddot{\varepsilon}_{\mathrm{vp}}(t)\right] + \left[\dot{\varepsilon}_{\mathrm{ve}}(t) + \dot{\varepsilon}_{\mathrm{vp}}(t)\right]\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}} + \frac{H'}{\lambda_{\mathrm{N}}}\right) + \left[\varepsilon_{\mathrm{ve}}(t) + \varepsilon_{\mathrm{vp}}(t)\right]\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}\\ &\quad = \dot{\sigma}(t)\left(\frac{1}{\lambda_{\mathrm{K}}} + \frac{1}{\lambda_{\mathrm{N}}}\right) + \sigma(t)\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}} + \frac{H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}\right) - \sigma_{\mathrm{Y}}\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}\right).\end{aligned} \tag{10}$$

Since:

$$\varepsilon_{\mathrm{ve}}(t) + \varepsilon_{\mathrm{vp}}(t) = \varepsilon(t) - \varepsilon_{\mathrm{e}}(t) = \varepsilon(t) - \frac{\sigma(t)}{E_{\mathrm{H}}},$$

$$\dot{\varepsilon}_{\mathrm{ve}}(t) + \dot{\varepsilon}_{\mathrm{vp}}(t) = \dot{\varepsilon}(t) - \dot{\varepsilon}_{\mathrm{e}}(t) = \dot{\varepsilon}(t) - \frac{\dot{\sigma}(t)}{E_{\mathrm{H}}},$$

$$\ddot{\varepsilon}_{\mathrm{ve}}(t)+\ddot{\varepsilon}_{\mathrm{vp}}(t)=\ddot{\varepsilon}(t)-\ddot{\varepsilon}_{\mathrm{e}}(t)=\ddot{\varepsilon}(t)-\frac{\ddot{\sigma}(t)}{E_{\mathrm{H}}}, \tag{11}$$

on the basis of Eqs. (9) and (10) we can form the following differential equation

$$\ddot{\varepsilon}(t)+\dot{\varepsilon}(t)\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}}+\frac{H'}{\lambda_{\mathrm{N}}}\right)+\varepsilon(t)\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}$$
$$=\frac{\ddot{\sigma}(t)}{E_{\mathrm{H}}}+\dot{\sigma}(t)\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}E_{\mathrm{H}}}+\frac{H'}{\lambda_{\mathrm{N}}E_{\mathrm{H}}}+\frac{1}{\lambda_{\mathrm{K}}}+\frac{1}{\lambda_{\mathrm{N}}}\right)+\sigma(t)\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}+\frac{H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}+\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}E_{\mathrm{H}}}\right)-\sigma_{\mathrm{Y}}\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}. \tag{12}$$

Let us examine this model in the case of creep when strain can appear freely, under a given load, which causes the constant stress σ_0. Eq. (12) will then be:

$$\ddot{\varepsilon}(t)+\dot{\varepsilon}(t)\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}}+\frac{H'}{\lambda_{\mathrm{N}}}\right)+\varepsilon(t)\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}=\sigma_0\left(\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}+\frac{H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}+\frac{E_{\mathrm{K}}H'}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}E_{\mathrm{H}}}\right)-\sigma_{\mathrm{Y}}\frac{E_{\mathrm{K}}}{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}} \tag{13}$$

Eq. (13) is inhomogeneous linear differential equation of the second order, whose general solution is

$$\varepsilon(t)=C_1\mathrm{e}^{\frac{-H't}{\lambda_{\mathrm{N}}}}+C_2\mathrm{e}^{\frac{-E_{\mathrm{K}}t}{\lambda_{\mathrm{K}}}}+\sigma_0\left(\frac{1}{H'}+\frac{1}{E_{\mathrm{K}}}+\frac{1}{E_{\mathrm{H}}}\right)-\sigma_{\mathrm{Y}}\frac{1}{H'}. \tag{14}$$

The constants C_1 and C_2 will be determined from the initial conditions: at $t = 0$,

$$\varepsilon_0=\frac{\sigma_0}{E_{\mathrm{H}}},$$

$$\dot{\varepsilon}_0=\sigma_0\left(\frac{1}{\lambda_{\mathrm{K}}}+\frac{1}{\lambda_{\mathrm{N}}}\right)-\sigma_{\mathrm{Y}}\frac{1}{\lambda_{\mathrm{N}}}. \tag{15}$$

The constants will then be:

$$C_1=-(\sigma_0-\sigma_{\mathrm{Y}})\frac{1}{H'},\qquad C_2=-\frac{\sigma_0}{E_{\mathrm{K}}}, \tag{16}$$

and the final solution is

$$\varepsilon(t)=\left(-\frac{(\sigma_0-\sigma_{\mathrm{Y}})}{H'}\right)\mathrm{e}^{\frac{-H't}{\lambda_{\mathrm{N}}}}+\left(-\frac{\sigma_0}{E_{\mathrm{K}}}\right)\mathrm{e}^{\frac{-E_{\mathrm{K}}t}{\lambda_{\mathrm{K}}}}+\sigma_0\left(\frac{1}{H'}+\frac{1}{E_{\mathrm{K}}}+\frac{1}{E_{\mathrm{H}}}\right)-\sigma_{\mathrm{Y}}\frac{1}{H'}. \tag{17}$$

When $t\rightarrow\infty$

$$\varepsilon(t)=\frac{\sigma_0-\sigma_{\mathrm{Y}}}{H'}+\sigma_0\left(\frac{1}{E_{\mathrm{H}}}+\frac{1}{E_{\mathrm{K}}}\right). \tag{18}$$

(see Fig. 2.)

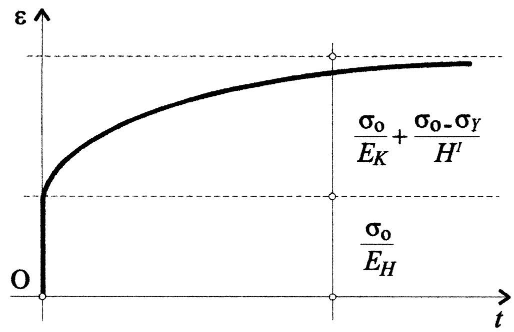


Fig. 2. Change of strain with time according to a new viscoelastoplastic body under constant stress σ_0.

3. Rheological–dynamical analogy (RDA)

3.1. Statement of the problem

The corresponding homogeneous equation of the inhomogeneous equation (12) has the following form:

$$\ddot{\varepsilon}(t)\lambda_K\lambda_N + \dot{\varepsilon}(t)\left(E_K\lambda_N + H'\lambda_K\right) + \varepsilon(t)E_KH' = 0, \tag{19}$$

where: λ_K, λ_N, E_K and H' are given constants at fixed step time.

$$[\lambda_K], [\lambda_N] = ML^{-1}T^{-1}, \qquad [E_K], [H'] = ML^{-1}T^{-2},$$

$$[\rho] = ML^{-3}, \qquad [g] = LT^{-2}, \qquad [\gamma] = [\rho g] = ML^{-2}T^{-2},$$

$$[\lambda_K\lambda_N] = M^2L^{-2}T^{-2} = M[\gamma] = [m\gamma],$$

$$\left[E_K\lambda_N + H'\lambda_K\right] = M^2L^{-2}T^{-3} = MT^{-1}[\gamma] = [c\gamma],$$

$$\left[E_KH'\right] = M^2L^{-2}T^{-4} = MT^{-2}[\gamma] = \left[k\gamma\right]. \tag{20}$$

Replacing $\lambda_K\lambda_N$ by $m \cdot \gamma$, $E_K \cdot \lambda_N + H' \cdot \lambda_K$ by $c \cdot \gamma$ and $E_K \cdot H'$ by $k \cdot \gamma$, the differential equation (19) becomes

$$\ddot{\varepsilon}(t)m + \dot{\varepsilon}(t)c + \varepsilon(t)k = 0, \tag{21}$$

where:

$$m = \frac{\lambda_K\lambda_N}{\gamma}, \qquad c = \frac{(E_K\lambda_N + H'\lambda_K)}{\gamma}, \qquad k = \frac{E_KH'}{\gamma}. \tag{22}$$

Analogy between differential equation (21) and differential equation of damped free vibration of single-degree-of-freedom (SDOF) system becomes obvious. Coefficients are dimensionally equal in these equations. Therefore, a very important new mathematical–physical analogy between the new rheological model and the dynamical model with viscous damping, Fig. 3, can be formulated. In accordance with

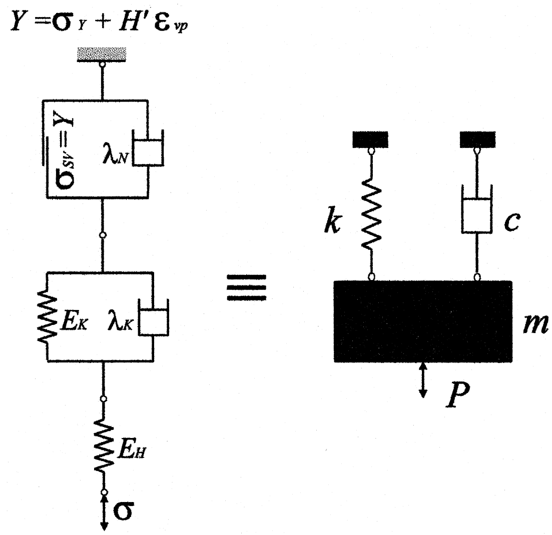


Fig. 3. Analogy between a new rheological model and dynamical model with viscous damping.

this it is obvious that inelastic response of engineering structures is essentially a dynamical problem. The rheology is a branch of physics, which is closest to mechanics. The analogy exists for a very specific rheological model, and is one example of numerous analogies that can be observed in mechanics, as well as between mechanical and electrical (thermal, magnetic, etc.) systems, by virtue of their mathematical descriptions.

3.2. Complementary solution

Next, Eq. (21) can be rewritten in the form

$$\ddot{\varepsilon}(t) + \left(\frac{c}{m}\right)\dot{\varepsilon}(t) + \left(\frac{k}{m}\right)\varepsilon(t) = 0, \tag{23}$$

or

$$\ddot{\varepsilon}(t) + 2\xi\dot{\varepsilon}(t) + \omega^2\varepsilon(t) = 0,$$

where:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{E_K H'}{\lambda_K \lambda_N}}, \qquad \frac{c}{2m} = \xi = \frac{E_K \lambda_N + H'\lambda_K}{2\lambda_K \lambda_N}. \tag{24}$$

For an equation of this type a solution of the form $\varepsilon = e^{rt}$ can be used. The values of r_1 and r_2 are found by taking the positive and negative signs in expression (25)

$$r_{1,2} = -\xi \pm \sqrt{\xi^2 - \omega^2}. \tag{25}$$

It is useful to consider separately the situations arising when the expression under the root sign in Eq. (25) is positive, negative or zero.

Table 1
Eigen function of strains

Overdamped case $\xi^2 > \omega^2$		Case of critical damping $\xi^2 = \omega^2 \wedge H' \geq 0$	
$\varepsilon(t) = C_1 \cdot \exp(r_1 \cdot t) + C_2 \cdot \exp(r_2 \cdot t),\ C_1 = (-\varepsilon_0 \cdot r_2 + \dot{\varepsilon}_0)/(r_1 - r_2),$ $C_2 = \varepsilon_0 - C_1$		$r_{1/2} = -\xi,\ \varepsilon(t) = (C_1 + C_2 \cdot t) \cdot \exp(-\xi \cdot t),\ C_1 = \varepsilon_0,$ $C_2 = \dot{\varepsilon}_0 + \xi \cdot \varepsilon_0$	
Strain-hardening material behavior (a) $H' > 0$	Strain-softening material behavior (e) $H' < 0$	Strain-hardening material behavior (b) $E_K \cdot \lambda_N = H' \cdot \lambda_K$	Ideal liquid (d) $E_K = H' = 0$
$r_1 = -H'/\lambda_N\ r_2 = -E_K/\lambda_K$	$r_1 = \lvert H' \rvert/\lambda_N\ r_2 = -E_K/\lambda_K$	$\xi = E_K/\lambda_K$	$\xi = 0$
(f) Undamped free strains, strain-softening material behavior ($H' < 0$)			
$c = \xi = 0,\ \omega_c = i \cdot \omega,\ r_{1/2} = \pm\omega, \varepsilon(t) = C_1 \cdot \exp(\omega \cdot t) + C_2 \cdot \exp(-\omega \cdot t),\ C_1 = (\varepsilon_0 \cdot \omega + \dot{\varepsilon}_0)/(2 \cdot \omega),\ C_2 = \varepsilon_0 - C_1$			
(c) Strains without strain-hardening or strain-softening material behavior ($H' = 0$)			
$\varepsilon(t) = C_1 + C_2 \cdot \exp(-E_K \cdot t/\lambda_K)\, C_1 = \varepsilon_0 + \dot{\varepsilon}_0(\lambda_K/E_K),\ C_2 = \varepsilon_0 - C_1$			

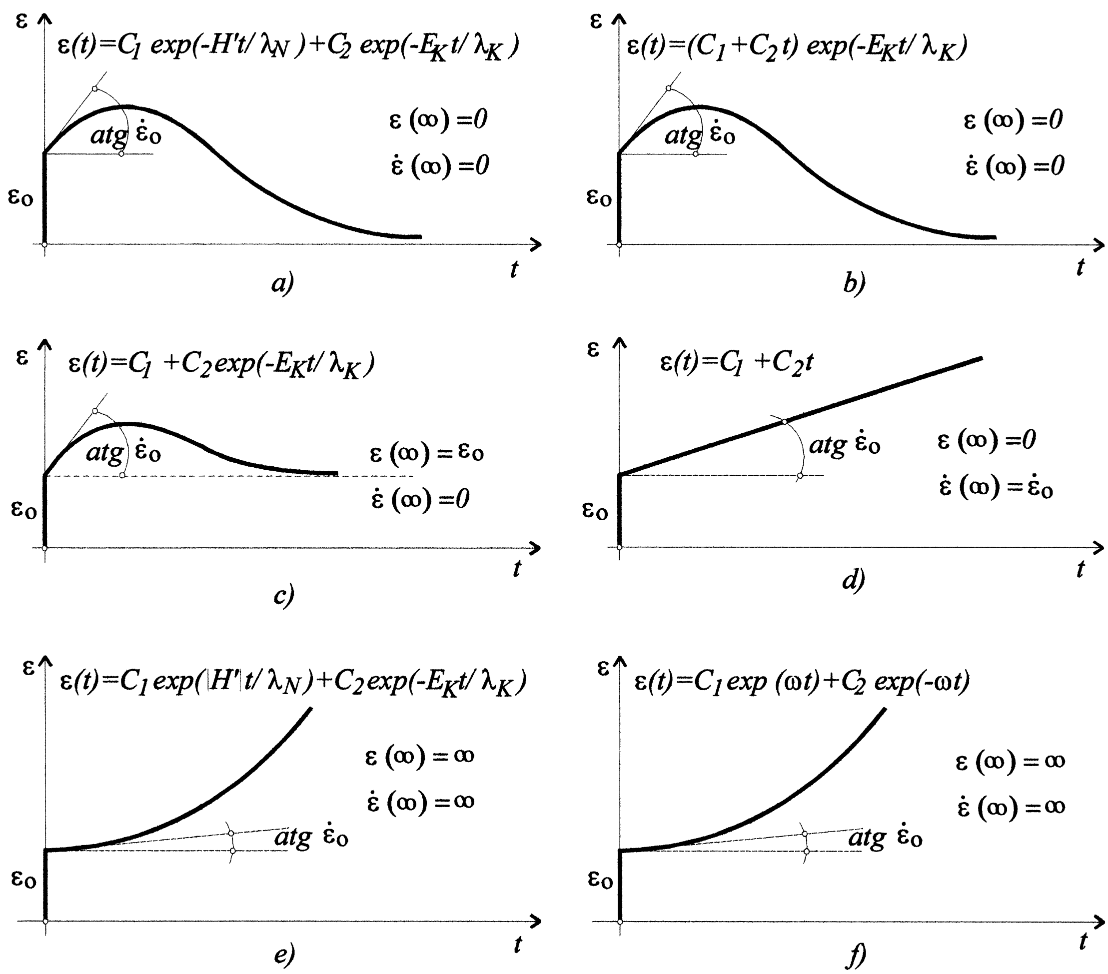


Fig. 4. Strain–time curves. (a) overdamped case, strain-hardening material behavior; (b) case of critical damping, strain-hardening material behavior; (c) strain without strain-hardening or strain-softening material behavior; (d) case of critical damping, ideal liquid; (e) overdamped case, strain-softening material behavior; (f) undamped free strain, strain-softening material behavior.

a) Overdamped case ($\xi^2 > \omega^2$).
b) Underdamped case ($\xi^2 < \omega^2$). The equation does not have its solution because mass, damping and stiffness are determined by Eq. (22).
c) Case of critical damping ($\xi^2 = \omega^2 \wedge H' \geq 0$).

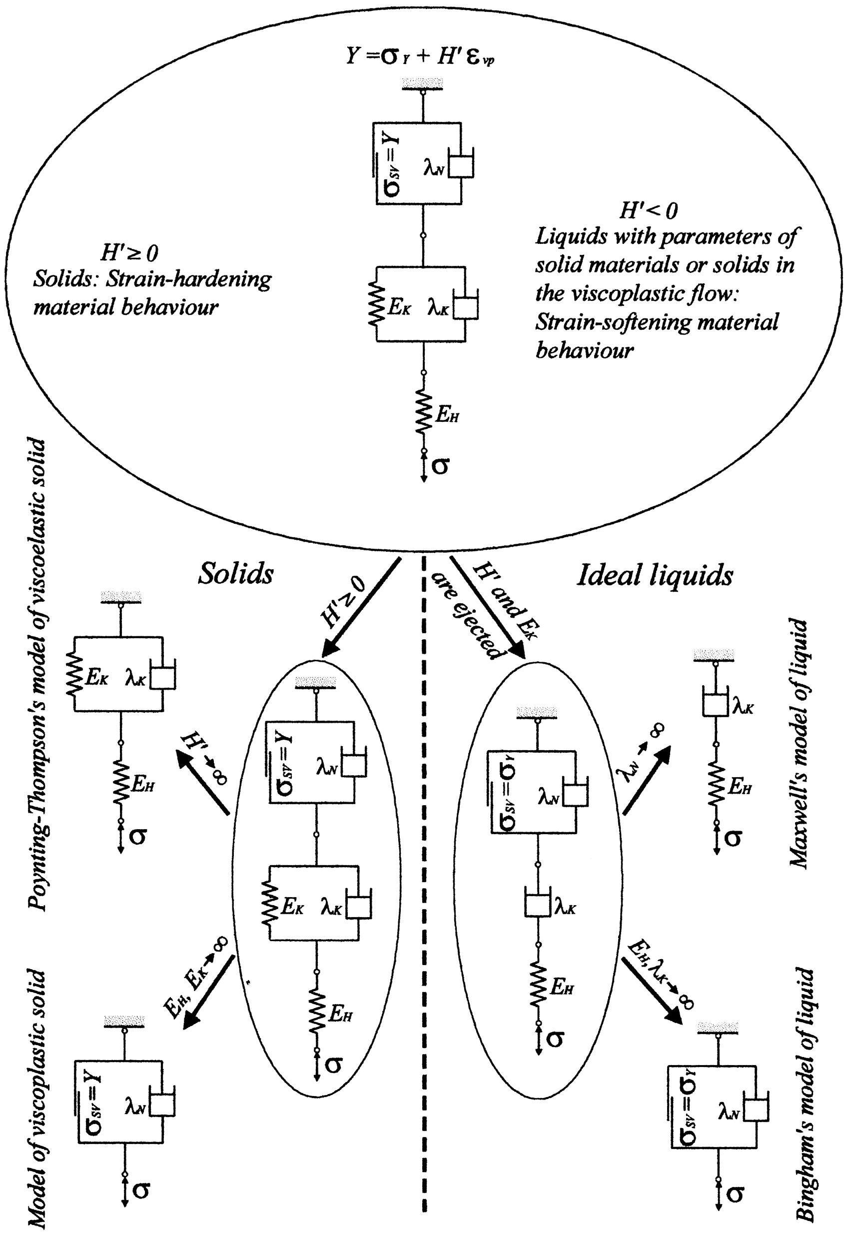


Fig. 5. A new rheological model, which includes solids and liquids, and in the case of the limit values of some parameters it decomposes to well-known rheological models.

Table 1 gives six eigen functions of strains (creep curves or flow functions). All solutions are shown in Fig. 4. This set of solutions represents (a), (b), (c), solids, (d) ideal liquid and (e), (f), liquids with parameters of solid material or solids in the viscoplastic yielding. However, it may be considered as an axiom of rheology that every real material possesses all rheological-properties, albeit in varying degrees, Fig. 5.

Consulting Fig. 5, we see how a system of rheological bodies is formed in accordance with the number of fundamental structural elements making up the body. Some possible combinations have been postulated either on theoretical grounds or from empirical considerations (Reiner, 1955). They are:

M = N–H named after MAXWELL (1868).
K = H|N as postulated by Lord KELVIN (1875).
Schw = H–(StV|M) = H–(StV|(N–H)) after SCHWEDOFF (1900).
PTh = H|M = H|(N–H) after POYNTING and THOMSON (1902).
TR = N–PTh = N–H|(N–H) as postulated by TROUTON and RANKINE (1904).
B = H–(StV|N) after BINGHAM (1919).
J = N|M = N|(N–H) after JEFFREYS (1929).
Bu = M–K = (N–H)–(H|N) after BURGERS (1935).
Sch ScB = Schw–K = H–(StV|(N–H))–(H|N) after SCHOFIELD and SCOTT BLAIR (1937).
L = K–N = (H|N)–N as postulated by LETHERSICH (1942).

3.3. Cyclic variation of stress

Cyclic variation of stress causes an increase of creep from Eq. (17), which is known as basic. We now consider uniaxial cyclic stress described by the sine function only for convenience, since cyclic creep does not depend much on the shape of the time curve within the cycle, Fig. 6.

$$\sigma(t) = \sigma_0 + \sigma_A \sin(\omega_\sigma t). \tag{26}$$

The rheological equation of the new model takes its most general form,

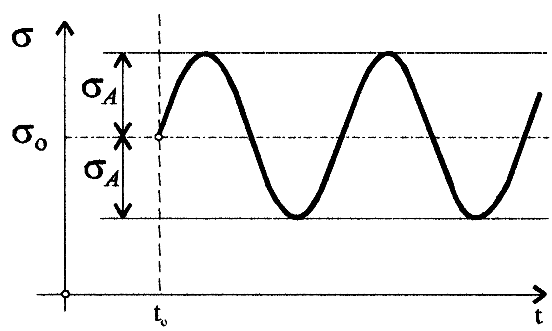


Fig. 6. Cyclic variation of stress.

$$\ddot{\varepsilon}(t)m + \dot{\varepsilon}(t)c + \varepsilon(t)k = \sigma_A\left(\frac{k}{E_H} + \frac{E_K + H'}{\gamma} - \omega_\sigma^2\frac{m}{E_H}\right)\sin(\omega_\sigma t) + \sigma_A\left(\frac{c}{E_H} + \frac{\lambda_K + \lambda_N}{\gamma}\right)\omega_\sigma\cos(\omega_\sigma t) + \sigma_0\left(\frac{k}{E_H} + \frac{E_K + H'}{\gamma}\right) - \sigma_Y\frac{E_K}{\gamma}. \tag{27}$$

where:σ_0 is constant stress, σ_A is cyclic stress amplitude, ω_σ is the load or stress frequency.

The solution for this second order differential equation with constant coefficients is

$$\varepsilon(t) = \varepsilon_h + \varepsilon_p, \tag{28}$$

where ε_h is called the complementary solution, and is given in Table 1. ε_p is the particular solution for the given equation.

$$\varepsilon_p = A\sin(\omega_\sigma t) + B\cos(\omega_\sigma t) + C, \tag{29}$$

where A, B and C are constants:

$$A = \frac{P_\sigma(k - m\omega_\sigma^2) + Q_\sigma c\omega_\sigma}{(k - m\omega_\sigma^2)^2 + (c\omega_\sigma)^2},$$

$$B = \frac{Q_\sigma(k - m\omega_\sigma^2) - P_\sigma c\omega_\sigma}{(k - m\omega_\sigma^2)^2 + (c\omega_\sigma)^2},$$

$$C = \sigma_0\left(\frac{1}{E_H} + \frac{1}{E_K} + \frac{1}{H'}\right) - \sigma_Y\frac{1}{H'}, \tag{30}$$

and:

$$P_\sigma = \sigma_A\left(\frac{k}{E_H} + \frac{E_K + H'}{\gamma}\right) - \sigma_A\omega_\sigma^2\frac{m}{E_H},$$

$$Q_\sigma = \sigma_A\left(\frac{c}{E_H} + \frac{\lambda_K + \lambda_N}{\gamma}\right)\omega_\sigma. \tag{31}$$

The cyclic strain is given by

$$\varepsilon_p''(t) = A\sin(\omega_\sigma t) + B\cos(\omega_\sigma t), \tag{32}$$

or

$$\varepsilon_p''(t) = \varepsilon_A\sin(\omega_\sigma t - \alpha), \tag{33}$$

where:

$$\varepsilon_A = \sqrt{\frac{P_\sigma^2 + Q_\sigma^2}{(k - m\omega_\sigma^2)^2 + (c\omega_\sigma)^2}},$$

$$\tan\alpha = \frac{P_\sigma c\omega_\sigma - Q_\sigma\left(k - m\omega_\sigma^2\right)}{P_\sigma\left(k - m\omega_\sigma^2\right) + Q_\sigma c\omega_\sigma}. \tag{34}$$

The periodic stress may be expressed by means of the exponential function

$$\sigma'' = \sigma_A e^{i\omega_\sigma t}. \tag{35}$$

The strain lagging behind the stress by the phase difference α is given by

$$\varepsilon'' = \varepsilon_A e^{i(\omega_\sigma t - \alpha)}. \tag{36}$$

The complex modulus may be expressed by the ratio of the variable stress to the variable strain as follows

$$E^* = \frac{\sigma''}{\varepsilon''} = \frac{\sigma_A}{\varepsilon_A} e^{i\alpha}. \tag{37}$$

According to the Moivre theorem, we have

$$e^{i\alpha} = \cos\alpha + i\sin\alpha, \tag{38}$$

and therefore Eq. (37) yields

$$E^* = \frac{\sigma_A}{\varepsilon_A}(\cos\alpha + i\sin\alpha), \tag{39}$$

where:

$$E_R = \operatorname{Re} E^* = \frac{\sigma_A}{\varepsilon_A}\cos\alpha, \qquad E_I = \operatorname{Im} E^* = \frac{\sigma_A}{\varepsilon_A}\sin\alpha. \tag{40}$$

Differentiating Eq. (36) with respect to time, we get the following expression for the strain rate

$$\dot{\varepsilon}'' = i\omega_\sigma \varepsilon_A e^{i(\omega_\sigma t - \alpha)}. \tag{41}$$

Dividing Eq. (35) by Eq. (41), we get the complex coefficient of viscosity

$$\lambda^* = \frac{\sigma''}{\dot{\varepsilon}''} = \frac{\sigma_A}{i\omega_\sigma \varepsilon_A} e^{i\alpha}. \tag{42}$$

Substituting from Eq. (38), we find

$$\lambda^* = \frac{\sigma_A}{i\omega_\sigma \varepsilon_A}(\cos\alpha + i\sin\alpha) = \frac{\sigma_A}{\omega_\sigma \varepsilon_A}(\sin\alpha - i\cos\alpha). \tag{43}$$

From the foregoing relation, we get formulae for the dynamic coefficient of viscosity and the loss coefficient of viscosity:

$$\lambda_R = \operatorname{Re}\lambda^* = \frac{\sigma_A}{\omega_\sigma \varepsilon_A}\sin\alpha, \qquad \lambda_I = \operatorname{Im}\lambda^* = \frac{\sigma_A}{\omega_\sigma \varepsilon_A}\cos\alpha. \tag{44}$$

The complex compliance is equal to the reciprocal of the complex modulus

$$I^* = \frac{1}{E^*} = \frac{1}{E_R + iE_I} = \frac{E_R - iE_I}{E_R^2 + E_I^2}. \tag{45}$$

For the dynamic and loss compliance, we obtain the relations:

$$I_{\mathrm{R}} = \mathrm{Re}\, I^* = \frac{E_{\mathrm{R}}}{E_{\mathrm{R}}^2 + E_{\mathrm{I}}^2} = \frac{\varepsilon_{\mathrm{A}}}{\sigma_{\mathrm{A}}}\cos\alpha,$$

$$I_{\mathrm{I}} = \mathrm{Im}\, I^* = \frac{E_{\mathrm{I}}}{E_{\mathrm{R}}^2 + E_{\mathrm{I}}^2} = \frac{\varepsilon_{\mathrm{A}}}{\sigma_{\mathrm{A}}}\sin\alpha. \tag{46}$$

Using Eq. (34) we find rheological–dynamic (RDA) modulus and RDA compliance:

$$E_{\mathrm{R}} = \sigma_{\mathrm{A}} \frac{P_\sigma\left(k - m\omega_\sigma^2\right) + Q_\sigma c\omega_\sigma}{P_\sigma^2 + Q_\sigma^2}, \tag{47}$$

$$I_{\mathrm{R}} = \frac{1}{\sigma_{\mathrm{A}}} \frac{P_\sigma\left(k - m\omega_\sigma^2\right) + Q_\sigma c\omega_\sigma}{\left(k - m\omega_\sigma^2\right)^2 + (c\omega_\sigma)^2}. \tag{48}$$

4. Buckling curves using RDA

4.1. General

Finally the concept, which can be applied for understanding of the phenomenon of in plane buckling, is introduced, and it is possible to identify the governing parameters and to present the procedures as a design method.

Generally speaking, the RDA is derived in order to solve the dynamical problems, but can also be used to explain any statical problems, considering the correspondent limit values of presented mathematical expressions.

When a slender column is loaded axially, there rheological behavior of column may be characterized by the time of retardation $T_{\mathrm{K}}^{\mathrm{D}}$. It may be calculated in a simple way

$$T_{\mathrm{K}}^{\mathrm{D}} = \sqrt{\frac{m}{k}} = \frac{1}{\omega} = \sqrt{\frac{\lambda_{\mathrm{K}}\lambda_{\mathrm{N}}}{E_{\mathrm{K}} H'}}, \tag{49}$$

where

$$\omega = \sqrt{\frac{E_{\mathrm{H}} g}{\gamma}} \frac{1}{L}. \tag{50}$$

Substituting Eq. (49) in Eq. (47) we obtain the RDA modulus in the following form

$$E_{\mathrm{R}}(t, t_0) = \frac{\dfrac{1+\delta^2}{E_{\mathrm{H}}(t_0)} + \dfrac{1}{E_{\mathrm{K}}^{\mathrm{D}}(t, t_0)} + \dfrac{1}{H'^{\mathrm{D}}(t, t_0)}}{\dfrac{1+\delta^2}{E_{\mathrm{H}}^2(t_0)} + \left(\dfrac{1}{E_{\mathrm{K}}^{\mathrm{D}}(t, t_0)} + \dfrac{1}{H'^{\mathrm{D}}(t, t_0)}\right)\left(\dfrac{2}{E_{\mathrm{H}}(t_0)} + \dfrac{1}{E_{\mathrm{K}}^{\mathrm{D}}(t, t_0)} + \dfrac{1}{H'^{\mathrm{D}}(t, t_0)}\right)}, \tag{51}$$

where

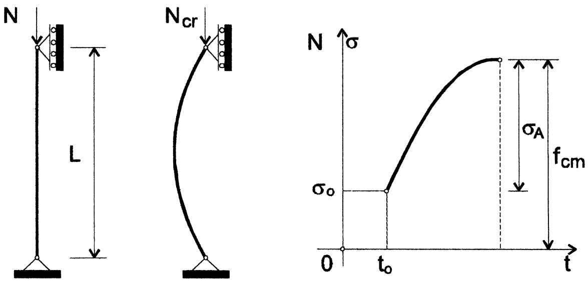


Fig. 7. Two-hinged column under time-varying loading.

$$\delta = \frac{\omega_\sigma}{\omega} = \omega_\sigma T_{\mathrm{K}}^{\mathrm{D}}. \tag{52}$$

There are two principal ways to design a compression member: the first involves a second order analysis whereby the equilibrium of moments and forces is calculated by considering the deformed shape of the respective member of structure. The second approach uses buckling curves to account the strength decrease in a real column in comparison to a compression member infinitely stiff in bending. Here, the stability design is carried out according to the second approach with modified compression strength using RDA modulus. The decrease in load-bearing capacity depends on the slenderness ratio of the member in question and is based on the behavior of a two-hinged column, see Fig. 7.

The factors influencing the load-bearing capacity of a column may be divided into two groups. The first group involves the nominal geometry of the compressed member such as: cross-section and length, its support conditions and the material properties which are determined by the choice of the strength class, the surrounding climate and the load duration class of the governing load case. The second group of factors also influencing column strength involves geometric and material imperfections and variations. Material imperfections include growth characteristics and other factors, which influence the stress–strain behavior of concrete, steel, timber, etc.

Generally, the stress–strain curve is linear elastic until $\sigma \le \sigma_{\mathrm{p}}$ (see Fig. 8) and non-linear with considerable viscoelastoplastic strain, under compression stress $\sigma \ge \sigma_{\mathrm{p}}$. Critical stresses are defined by:

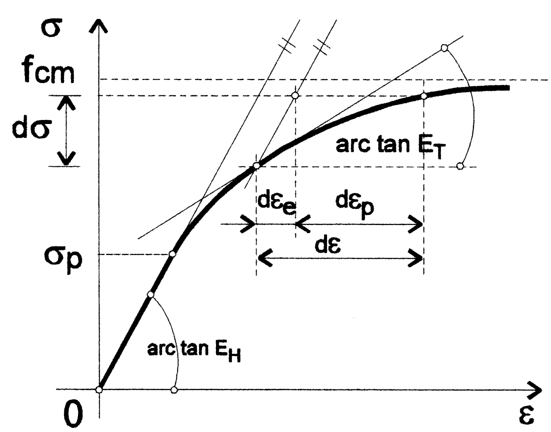


Fig. 8. Stress–strain curve: f_{cm} is the value of compression strength.

$$\sigma_{cr} = \begin{cases} \dfrac{\pi^2}{(L/k_z)^2} E_H(t_0) & \text{for } \sigma_{cr} \le \sigma_p, \\ E_R(t, t_0) E_H(t_0) & \text{for } \sigma_{cr} \ge \sigma_p \end{cases}. \tag{53}$$

From Fig. 8, the slope of the plastic strain is

$$H'(\varepsilon_p) = \frac{d\sigma}{d\varepsilon_p} = \frac{d\sigma}{d\varepsilon - d\varepsilon_e} = \frac{E_T}{1 - \dfrac{E_T}{E_H}}. \tag{54}$$

In the special case, when $\delta \to 0$, we have statical loading and RDA modulus from Eq. (51) as follows

$$E_R(t, t_0) = \frac{\dfrac{1}{E_H(t_0)} + \dfrac{1}{E_K^D(t, t_0)} + \dfrac{1}{H'^D(t, t_0)}}{\dfrac{1}{E_H^2(t_0)} + \left(\dfrac{1}{E_K^D(t, t_0)} + \dfrac{1}{H'^D(t, t_0)}\right)\left(\dfrac{2}{E_H(t_0)} + \dfrac{1}{E_K^D(t, t_0)} + \dfrac{1}{H'^D(t, t_0)}\right)}$$

$$= \frac{1}{\left(\dfrac{1}{E_H(t_0)} + \dfrac{1}{E_K^D(t, t_0)} + \dfrac{1}{H'^D(t, t_0)}\right)}. \tag{55}$$

In this way the dynamical problem is reduced to statical one.

If the solid has not viscoelastic strain, it has viscoplastic behavior, where $E_K^D(t, t_0) \to \infty$. The boundary value of the Eq. (55) is

$$E_R(t, t_0) = \frac{1}{\dfrac{1}{E_H(t_0)} + \dfrac{1}{H'^D(t, t_0)}} = \frac{E_H(t_0) H'^D(t, t_0)}{E_H(t_0) + H'^D(t, t_0)} = \frac{H'^D(t, t_0)}{1 + \dfrac{H'^D(t, t_0)}{E_H(t_0)}}. \tag{56}$$

which gives

$$H'^D(t, t_0) = \frac{E_R(t, t_0)}{1 - \dfrac{E_R(t, t_0)}{E_H(t_0)}}. \tag{57}$$

Comparing Eqs. (54) and (57), we get that RDA modulus is equal to the tangent modulus from the curve in Fig. 8.

$$E_R(t, t_0) = E_T. \tag{58}$$

4.2. Buckling curves of columns using RDA

A typical creep curve of the majority of materials, corresponding to constant stress and constant temperature, is given in Fig. 9. Strain starts with the initial, instantaneous strain ε_0, and increases with time. In general there are three ranges of creep: primary, secondary and tertiary ranges, as indicated in Fig. 9. The secondary creep regime is generally considered as the period of stationary creep conditions with a constant creep strain rate.

Creep curves are strongly dependent on stress level and on temperatures. For a given temperature creep strain increases with stress, and for a given stress, creep increases with temperature. We consider

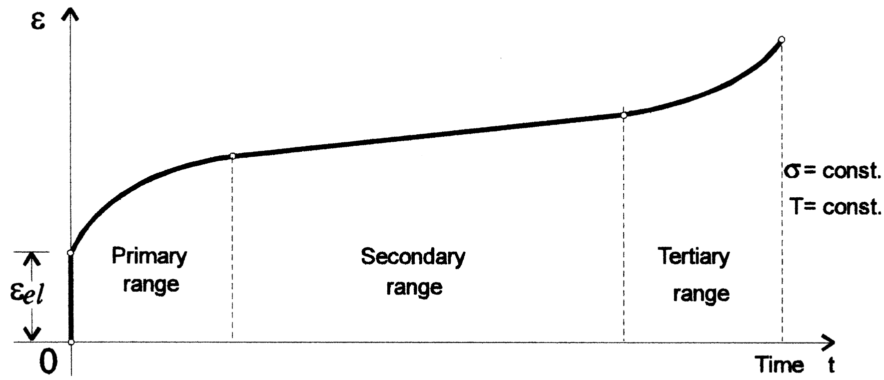


Fig. 9. Creep curve of the majority of materials.

here uniaxial creep law of steel, known as power creep law in the period of stationary creep conditions (Kojiç, 1997).

$$\varepsilon^{c} = a_0\sigma^{a_1}t^{a_2}, \tag{59}$$

where a_0, a_1 and a_2 are material constants. Now, let us consider uniaxial creep from Eq. (17), known as viscoelastic creep

$$\varepsilon^{c} = \sigma_0\frac{1}{E_K}\left(1 - e^{-\frac{E_K}{\lambda_K}t}\right), \tag{60}$$

and ratio

$$\varphi(t) = \frac{\varepsilon^{c}}{\varepsilon_{el}} = \frac{E_H(t_0)}{E_K}\left(1 - e^{-\frac{E_K}{\lambda_K}t}\right), \tag{61}$$

from which we obtain

$$E_K = \frac{E_H(t_0)}{\varphi(t)}\left(1 - e^{-\frac{E_K}{\lambda_K}t}\right), \tag{62}$$

where

$$\lambda_K = E_K T_K. \tag{63}$$

In order to apply law (59) we need to determine ratio

$$\varphi(t) = \frac{\varepsilon^{c}}{\varepsilon_{el}} = \frac{a_0\sigma^{a_1}t^{a_2}}{\dfrac{\sigma}{E_H(t_0)}} = E_H(t_0)a_0\sigma^{a_1-1}t^{a_2}. \tag{64}$$

Now, taking into account Eqs. (49) and (63) we have RDA modulus

$$E_K^{D}(t, t_0) = \frac{E_H(t_0)}{\varphi(t)}\left(1 - e^{-\frac{t}{T_K^{D}}}\right). \tag{65}$$

Further, using Eq. (50)

$$E_{\mathrm{K}}^{\mathrm{D}}(t,t_0)=\frac{E_{\mathrm{H}}(t_0)}{\varphi(t)}(1-\mathrm{e}^{-\omega t})=\frac{E_{\mathrm{H}}(t_0)}{\varphi(t)}. \tag{66}$$

where

$$\mathrm{e}^{-\omega t}=\mathrm{e}^{-\sqrt{\frac{E_{\mathrm{H}}g}{\gamma}}\frac{1}{L}t}\approx 0.$$

With known $E_{\mathrm{K}}^{\mathrm{D}}(t,t_0)$ we determine $H'^{\,\mathrm{D}}(t,t_0)$ from Eq. (22) as

$$H'^{\,\mathrm{D}}(t,t_0)=\frac{k\gamma}{E_{\mathrm{H}}(t_0)}\varphi(t)=\frac{E_{\mathrm{H}}(t_0)A\gamma}{LE_{\mathrm{H}}(t_0)}\varphi(t). \tag{67}$$

Now we write RDA modulus

$$E_{\mathrm{R}}(t,t_0)=\frac{1}{\left(\frac{1}{E_{\mathrm{H}}(t_0)}+\frac{\varphi(t)}{E_{\mathrm{H}}(t_0)}+\frac{L}{A\gamma\varphi(t)}\right)}. \tag{68}$$

In calculation the viscoplastic deforming according RDA, the elastic and viscoelastic strains are very small values ($\frac{1}{E_{\mathrm{H}}(t_0)}\approx 0$ and $\frac{\varphi(t_0)}{E_{\mathrm{H}}(t_0)}\approx 0$), and we have from Eq. (68)

$$E_{\mathrm{R}}(t,t_0)=\frac{1}{\frac{L}{A\gamma\varphi(t)}}, \tag{69}$$

or

$$E_{\mathrm{R}}(t,t_0)=\frac{1}{\frac{I_z}{A}\frac{L}{I_z}\frac{1}{\gamma\varphi(t)}}=\frac{1}{k_z^2\frac{L}{I_z}\frac{1}{\gamma\varphi(t)}}=\frac{1}{\left(\frac{L}{k_z}\right)\frac{k_z^3}{I_z}\frac{1}{\gamma\varphi(t)}}. \tag{70}$$

Buckling curves in Eq. (71) generally describe the influence of slenderness on the characteristic load-bearing capacity of two-hinged columns.

$$\sigma_{\mathrm{cr}}=\begin{cases}\dfrac{\pi^2}{\left(\frac{L}{k_z}\right)^2}E_{\mathrm{H}}(t_0) & \text{for } \sigma_{\mathrm{cr}}\le\sigma_{\mathrm{p}},\\[2ex] \dfrac{1}{\left(\frac{L}{k_z}\right)\frac{k_z^3}{I_z}\frac{1}{\gamma\varphi(t)}}E_{\mathrm{H}}(t_0) & \text{for } \sigma_{\mathrm{cr}}\ge\sigma_{\mathrm{p}}\end{cases}. \tag{71}$$

Each value on a buckling curve consequently represents the characteristic load-bearing capacity of columns with the corresponding slenderness ratio. The slenderness ratio is defined as the largest ratio of the unbraced length to the radius of gyration.

Eq. (71) allows determining the intersection (72) of Euler's and RDA curves, what is important for determination of lower (Euler's) and upper (RDA) slenderness boundary values, for which the curve equations are satisfied.

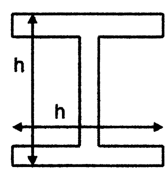
$$\left(\frac{L}{k_z}\right) = \pi^2 \frac{k_z^3}{I_z} \frac{1}{\gamma\varphi(t)}. \tag{72}$$

5. Numerical examples

5.1. Prediction of buckling curves of steel columns

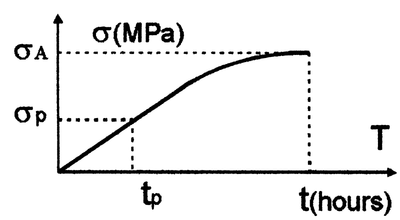
The following numerical examples illustrate the process of prediction of buckling curves using the RDA modulus. As a first, let us consider the two typical steel columns.

Example 1. Determine buckling curve of steel column with dimensions and cross-section as follow (Timoshenko, 1958) and compare the RDA solution with the Euler's solution.



$h = 140$ mm, $L = 180$ cm, $I_z = 64.4$ cm^4, $k_z = 1.73$, $I_y = 712$ cm^4, $A = 21.5$ cm^2, $\frac{k_z^3}{I_z} = 0.08$

Loading data:



$\sigma(t) = \sigma_A \cdot \sin(\omega_\sigma \cdot t)$, $\sigma_A = 360$ MPa, $\omega_\sigma = \frac{\pi}{2T}$, $T = 18$ min, $\sigma_p \approx 0.8 \times 240 \approx 191.5$ MPa.

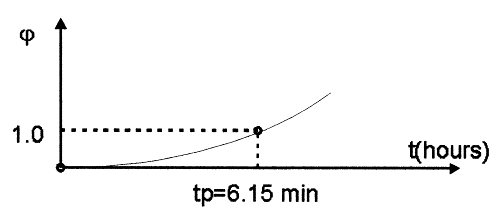
Material isotropic:

elastic modulus, $E = 2.1 \times 10^5$ MPa.

specific gravity, $\gamma = 7.86 \times 10^{-3}$ kg/cm^3.

creep coefficient, $\varphi = 1.0$.

Creep data (Kojić, 1997):



$\varepsilon^c = a_0 \cdot \sigma^{a_1} \cdot t^{a_2}$, $a_0 = 2.0 \times 10^{-9}$, $a_1 = 3.0$, $a_2 = 1.2$,

$$\varphi(t) = 2 \times 10^{-9} \cdot \sigma^2 \cdot t^{1.2} \cdot E_{\mathrm{H}},$$

$$\varphi(t_{\mathrm{p}}) = 2 \times 10^{-9} \times 191.5^2 \times \left(\frac{t_{\mathrm{p}}}{60}\right)^{1.2} \times 2.1 \times 10^5 = 1.0 \Longrightarrow t_{\mathrm{p}} = 6.15\mathrm{min}.$$

$$\sigma_{\mathrm{cr}}^{\mathrm{RDA}} = \frac{E}{\frac{L}{k_z} \cdot \frac{k_z^3}{I_z} \cdot \frac{1}{\gamma \cdot \varphi}} = \frac{210{,}000 \times 7.86 \times 10^{-3} \times 1.0}{\frac{L}{k_z} \cdot 0.08}.$$

The results of this expression are given in Table 2 and shown in Fig. 10.

$$\left(\frac{L}{k_z}\right) = \pi^2 \times 0.08 \times \frac{1}{7.86 \times 10^{-3} \times 1.0} = 100.45$$

The intersection of Euler's and RDA curves is obtained by adopted creep coefficient, $\varphi = 1.0$. Euler's and RDA curves intersection on slenderness ratio $\lambda = 100$ is confirmed by numerous experimentally obtained results.

Example 2. Determine buckling curve of steel column with dimensions and cross-section as follow, and compare the RDA solution with the tangent-modulus formula.

Table 2
Critical stresses of steel columns

L (cm)	λ	σ_{cr}^{E} (MPa)	$\sigma_{\mathrm{cr}}^{\mathrm{RDA}}$ (MPa)
40	23.12	3876.96	892.36
50	28.90	2481.25	713.88
60	34.68	1723.09	594.90
70	40.46	1265.95	509.92
80	46.24	969.24	446.18
90	52.02	765.82	396.60
100	57.80	620.31	356.94
110	63.58	512.66	324.49
120	69.36	430.77	297.45
130	75.14	367.05	274.57
140	80.92	316.49	254.96
150	86.71	275.69	237.96
160	92.49	242.31	223.09
170	98.27	214.64	209.97
180	104.05	191.45	198.30
190	109.83	171.83	187.86
200	115.61	155.08	178.47
210	121.39	140.66	169.97
220	127.17	128.16	162.25
230	132.95	117.26	155.19
240	138.73	107.69	148.73
250	144.51	99.25	142.78
260	150.29	91.76	137.29
270	156.07	85.09	132.20

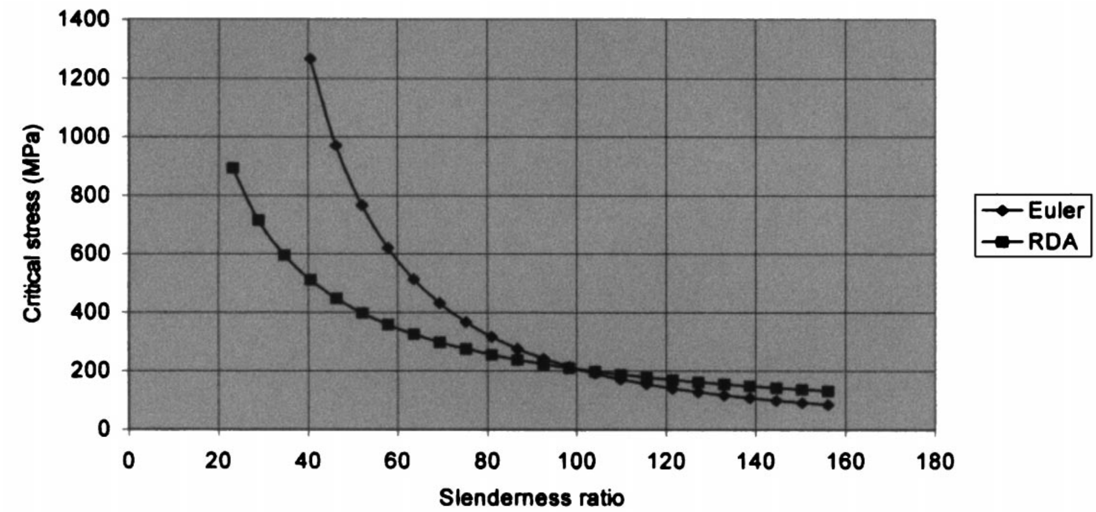
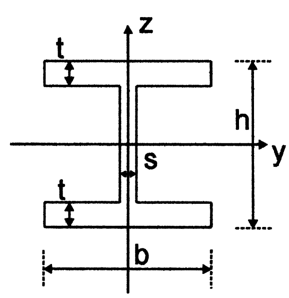


Fig. 10. Prediction of buckling curves of steel column using RDA and Euler's formula.

In this example, the data of Smith and Sidebottom (1965) are used for their comparisons with RDA results.

Consider an H-section which has a depth h, a width $b = h$, a flange thickness $t = h/16$, and a web thickness $t_s = h/24$. These relative dimensions are nearly identical with some rolled H-sections. Material has a yield stress, $\sigma_Y = 36{,}000$ psi (248.22 MPa, 1 psi = 6895 Pa). Beedle (1960) has shown that tangent-modulus theory can be used to design centrally loaded columns. Determine buckling curve of the column stub with dimensions:



$h = 40.64$ cm, $b = 40.64$ cm, $t = 2.54$ cm, $s = 1.69$ cm, $L = 1219$ cm, $A = 266.20$ cm^2; $I_y = 82574.91$ cm^4; $I_z = 28731.82$ cm^4; $k_z = \sqrt{I_z/A} = 10.389$; $k_z^3/I_z = 0.039$.

Loading data, material isotropic and creep data are specified in the Example 1. As indicated in Smith and Sidebottom (1965), is the fact that elastic (Euler's) theory is not valid for slenderness ratio under the 104.

$$\left(\frac{L}{k_z}\right) = \pi^2 \cdot \frac{k_z^3}{I_z} \cdot \frac{1}{\gamma \cdot \varphi(t_p)} = 104,$$

$$\varphi(t_p) = \pi^2 \times 0.039 \times \frac{1}{7.86 \times 10^{-3} \times 104} = 0.470878$$

$$\varphi(t_p) = 2 \times 10^{-9} \times 191.5^2 \times \left(\frac{t_p}{60}\right)^{1.2} \times 2.1 \times 10^5 = 0.470878$$

$$\Longrightarrow t_p = 3.3 \text{ min}$$

$$\sigma_{cr}^{RDA} = \frac{E}{\frac{L}{k_z} \cdot \frac{k_z^3}{I_z} \cdot \frac{1}{\gamma \cdot \varphi}} = \frac{210000 \times 7.86 \times 10^{-3} \times 0.470878}{\frac{L}{k_z} \cdot 0.039}$$

If we choose $t_p = 3.0$ min for our investigation we get:

$$\varphi(3.0) = 2 \times 10^{-9} \times 191.5^2 \times \left(\frac{3}{60}\right)^{1.2} \times 2.1 \times 10^5 = 0.42301,$$

$$\sigma_{cr}^{RDA} = \frac{E}{\frac{L}{k_z} \cdot \frac{k_z^3}{I_z} \cdot \frac{1}{\gamma \cdot \varphi}} = \frac{210000 \times 7.86 \times 10^{-3} \times 0.42301}{\frac{L}{k_z} \times 0.039}.$$

Table 3
Critical stresses of steel columns

L (cm)	λ	σ_{cr}^{E} (MPa)	σ_{cr}^{RDA} (t = 3.3 min) (MPa)	σ_{cr}^{RDA} (t = 3.0 min) (MPa)	Tangent modulus formula (Smith and Sidebottom, 1965)
207.78	20	5181.54	996.45	895.15	
311.67	30	2302.91	664.30	596.77	256.15
415.56	40	1295.39	498.23	447.58	256.00
519.45	50	829.05	398.58	358.06	252.00
623.34	60	575.73	332.15	298.38	244.77
727.23	70	422.98	284.70	255.76	228.00
831.12	80	323.85	249.11	223.79	212.00
935.01	90	255.88	221.43	198.92	201.50
1038.90	100	207.26	199.29	179.03	193.00
1059.68	102	199.21	195.38	175.52	192.00
1080.46	104	191.63	191.63	172.15	190.99
1101.23	106	184.46	188.01	168.90	
1122.01	108	177.69	184.53	165.77	
1142.79	110	171.29	181.17	162.76	
1163.57	112	165.23	177.94	159.85	
1184.35	114	159.48	174.82	157.04	
1205.12	116	154.03	171.80	154.34	
1225.90	118	148.85	168.89	151.72	
1246.68	120	143.93	166.08	149.19	
1350.57	130	122.64	153.30	137.72	
1454.46	140	105.75	142.35	127.88	
1558.35	150	92.12	132.86	119.35	
1662.24	160	80.96	124.56	111.89	
1766.13	170	71.72	117.23	105.31	
1870.02	180	63.97	110.72	99.46	

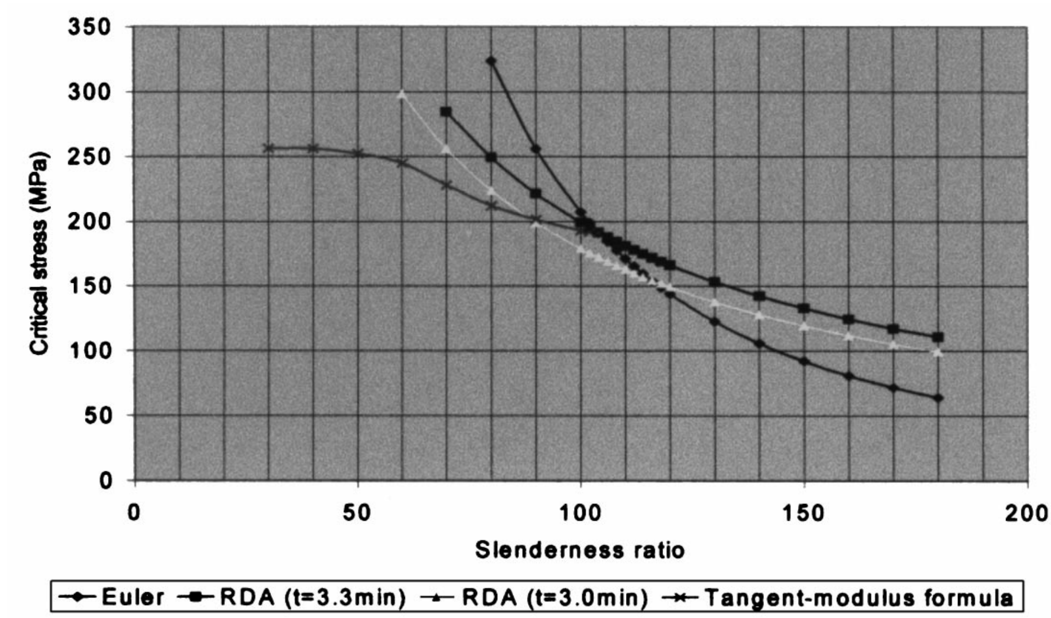


Fig. 11. Comparisons of the RDA solutions, with data of Smith and Sidebottom (1965) (tangent-modulus formula).

Comparisons of the RDA solutions with data of Smith and Sidebottom (1965) are given in the Table 3 and shown in Fig. 11. Different loading times are indicated on each curve.

Results are shown for two different loading times; they compare reasonable well with results obtained by tangent-modulus theory.

5.2. Prediction of buckling curves of wooden columns

Let us analyze the typical wooden column (Blass, 1995) in order to compare the results obtained by RDA and EC5.

Example 3.

5.2.1. EC5 characteristic and design buckling strengths

Determine the buckling curves of timber column with square cross-section 200 × 200 mm, buckling length of $L = 4.0$ m, and of timber strength class C24 according to prEN 338.

- compressive strength: $f_{c,0,k} = 21\ \text{N/mm}^2$
- modulus of elasticity: $E_{0,05} = 7400\ \text{N/mm}^2$
- factor taking into account the initial imperfection (solid timber): $\beta_c = 0.2$
- service class: $k_{mod} = 0.9$

5.2.1.1. Characteristic buckling strength according to EC5: 1995-1-1.

$$\sigma_{\mathrm{c,\ crit}} = \sigma_{\mathrm{Euler}} = \frac{\pi^2 \cdot E_{0,\ 05}}{\lambda^2}$$

$$\lambda_{\mathrm{rel}} = \sqrt{\frac{f_{c,\ 0,\ k}}{\sigma_{\mathrm{c,\ crit}}}}$$

$$k_{\mathrm{c}} = \frac{1}{k + \sqrt{k^2 - \lambda_{\mathrm{rel}}^2}}$$

$$k = 0.5\left[1 + \beta_{\mathrm{c}}(\lambda_{\mathrm{rel}} - 0.5) + \lambda_{\mathrm{rel}}^2\right]$$

Characteristic buckling strengths ($k_{\mathrm{c}} \cdot f_{c,\ 0,\ k}$) are given in the Table 4, for $\lambda = 10, 20, \ldots, 200$.
For the column adopted in the example, the following values are obtained:

$$A = 20^2 = 400\ \mathrm{cm}^2$$

$$I_y = \frac{20^4}{12} = 13333.3\dot{3}\ \mathrm{cm}^4$$

Table 4
Critical stresses of wooden columns

λ	σ_{cr}^{E} (MPa)	λ_{rel}	k	k_{c}	$k_{\mathrm{c}} \cdot f_{c,\ 0,\ k}$ (MPa)	$k_{\mathrm{c}} \cdot f_{c,\ 0,\ d}$ (MPa)	$\sigma_{\mathrm{cr}}^{\mathrm{RDA}}$ (MPa)
10	730.35	0.17	0.48	1.07	22.54	15.56	73.03
20	182.59	0.34	0.54	1.04	21.80	15.05	36.52
30	81.15	0.51	0.63	1.00	20.95	14.47	24.34
40	45.65	0.68	0.75	0.94	19.76	13.64	18.26
50	29.21	0.85	0.89	0.85	17.82	12.30	14.61
60	20.29	1.02	1.07	0.72	15.02	10.37	12.17
70	14.91	1.19	1.27	0.58	12.11	8.36	10.43
80	11.41	1.36	1.51	0.46	9.73	6.72	9.13
90	9.02	1.53	1.77	0.38	7.90	5.46	8.11
100	7.30	1.70	2.06	0.31	6.52	4.50	7.30
110	6.04	1.87	2.38	0.26	5.46	3.77	6.64
120	5.07	2.03	2.72	0.22	4.63	3.20	6.09
130	4.32	2.20	3.10	0.19	3.98	2.75	5.62
140	3.73	2.37	3.51	0.16	3.45	2.38	5.22
150	3.25	2.54	3.94	0.14	3.02	2.09	4.87
160	2.85	2.71	4.40	0.13	2.67	1.84	4.56
170	2.53	2.88	4.89	0.11	2.37	1.64	4.30
180	2.25	3.05	5.41	0.10	2.12	1.47	4.06
190	2.02	3.22	5.96	0.09	1.91	1.32	3.84
200	1.83	3.39	6.54	0.08	1.73	1.20	3.65

$$k_y = \sqrt{\frac{I_y}{A}} = 5.7735$$

$$\lambda_y = \frac{l}{k_y} = \frac{400}{5.7735} = 69$$

$$\sigma_{c,\ crit} = \sigma_{Euler} = \frac{\pi^2 \cdot E_{0.05}}{\lambda^2} = \pi^2 \cdot \frac{7400}{69^2} = 15.3\ \mathrm{N/mm^2}$$

$$\lambda_{rel} = \sqrt{\frac{f_{c,\ 0,\ k}}{\sigma_{c,\ crit}}} = \sqrt{\frac{21}{15.3}} = 1.17$$

$$k = 0.5\left[1 + \beta_c(\lambda_{rel} - 0.5) + \lambda_{rel}^2\right] = 0.5\left[1 + 0.2 \cdot (1.17 - 0.5) + 0.17^2\right] = 1.25$$

$$k_c = \frac{1}{k + \sqrt{k^2 - \lambda_{rel}^2}} = \frac{1}{1.25 + \sqrt{1.25^2 - 1.17^2}} = 0.59$$

$$k_c \cdot f_{c,\ 0,\ k} = 0.59 \times 21 = 12.39\ \mathrm{N/mm^2}$$

5.2.1.2. Design values of buckling strengths.

$$f_{c,\ 0,\ d} = \frac{k_{mod} \cdot f_{c,\ 0,\ k}}{\gamma_M} = \frac{0.9 \times 21}{1.3} = 14.5\ \mathrm{N/mm^2}$$

Design buckling strength values ($k_c \cdot f_{c,\ 0,\ d}$) are given in the Table 4, for $\lambda = 10, 20, \ldots, 200$.

In the case of column adopted for the analysis in this example, the following value is obtained:

$$k_c \cdot f_{c,\ 0,\ d} = 0.59 \times 14.5 = 8.56\ \mathrm{N/mm^2}.$$

5.2.2. RDA buckling curve

Determine the buckling curve according to RDA for the adopted strength class C24:

$$\gamma = 0.38 \times 10^{-3}\ \mathrm{kg/cm^3}, \qquad \frac{k_y^3}{I_y} = 0.0144.$$

The intersection of Euler's and RDA curves is adopted on slenderness ratio 100, what is at the same time the lower boundary of elastic domain on which the proportional stress σ_p is determined.

$$\lambda = \frac{l}{k_y} = \pi^2 \cdot \frac{k_y^3}{I_y} \cdot \frac{1}{\gamma \cdot \varphi} = 100$$

$$\varphi = \pi^2 \times 0.0144 \cdot \frac{1}{0.38 \times 10^{-3} \times 100} = 3.7488.$$

Choice of Euler's and RDA curves intersection on slenderness ratio $\lambda = 100$ can be explained by the fact that elastic (Euler's) theory is not valid for slenderness ratio under the 100, what is confirmed by numerous experimentally obtained results. In the range between $\lambda = 40$ and $\lambda = 100$, the progress of the plastic deformability of the timber under compressive stress has the most favorable effect (Blass, 1991).

Rheological–dynamical analogy is derived for studying the viscoelastoplastic problems, understanding that primer condition for its use is elastic solution. In this example, with choose $\lambda = 100$ for intersection of Euler's elastic and RDA curves, it is understood that RDA is valid for slenderness ratio $\lambda \leq 100$.

Specified intersection $\lambda = 100$ of Euler's elastic buckling and RDA curves for inelastic (viscoplastic) deformations is used in calculation of creep coefficient $\varphi = 3.7488$ and further in RDA expression:

$$\sigma_{\mathrm{cr}} = \frac{1}{\dfrac{l}{k_y} \cdot \dfrac{k_y^3}{I_y} \cdot \dfrac{1}{\gamma \cdot \varphi}} \cdot E_{0.05} = \frac{7400 \times 0.38 \times 10^{-3} \times 3.7488}{\dfrac{l}{k_y} \cdot 0.0144}.$$

The results of this expression are given in the Table 4, for $\lambda = 10, 20, \ldots, 200$, and shown in Fig. 12. Besides the numerous experiments in timber with the aim to determine the creep coefficient: under the long-term and short-term loading, low and high stress levels, monotone and cyclic stress variation, under the constant and changed environmental conditions (primary relative humidity), etc., the explicit values of this coefficient for the buckling curves are not given in EC5. However, this coefficient is taking into account in expressions given in Section 5.2.1, through β_c, k, and k_c (EC5). In calculation the inelastic deforming according RDA, the basic parameters are: creep coefficient (calculated in intersection point of Euler's and RDA curves at $\lambda = 100$), specific gravity and modulus of elasticity $E_{0.05}$ for correspondent timber strength class. Because of indirect introduction of creep coefficient in EC5, the discussion about this coefficient and its comparison with RDA procedure is possible only in the sense of buckling curves comparison.

Diagrams Fig. 12 represents EC5 characteristic and design buckling strengths, RDA inelastic (viscoplastic) deforming and Euler's elastic deforming. Undoubtedly, the comparison of these curves is possible in the range $\lambda \leq 100$, as in the theoretical sense having in mind the derived expressions, as well as in the practical sense from the aspects of their more or less easy use in concrete tasks. The agreement of the obtained results lays in the fact that Euler's elastic and RDA viscoplastic curves are envelopes for

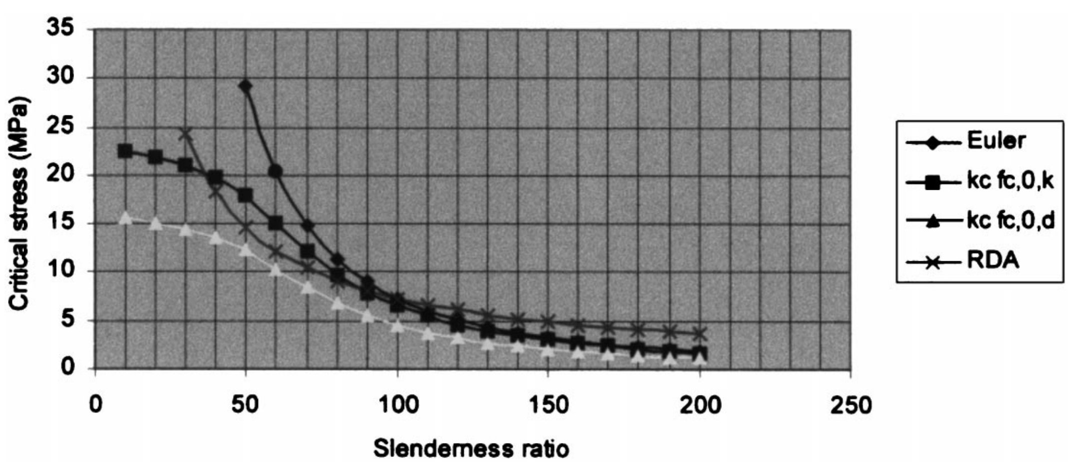


Fig. 12. EC5 characteristic and design buckling strengths, RDA inelastic buckling and Euler's elastic buckling curves.

EC5 characteristic and design curves. In theoretical and practical sense, the RDA solution simplifies the procedure and gives quite acceptable results.

5.3. Prediction of buckling curve of concrete columns

Example 4. To illustrate buckling curve of concrete columns, let us take the data of Hirst and Neville (1977). Prisms 7.6 cm × 7.6 cm, $L = 20.3, 30, \ldots, 450$ cm, fog-cured at (20 ± 1)°C, are loaded at the age of 28 days and allowed to dry at 50% relative humidity. 28-day strength of prisms is $f_{cm} = 46.5$ MPa. Specific gravity is $\gamma = 2.5 \times 10^{-3}$ kg/cm^3. Rapid-hardening portland cement; water : cement : sand : coarse aggregate ratio = 0.5 : 1 : 2 : 4. Initial elastic strains assumed to be 3.25×10^{-11} Pa^{-1} = $1/E$ ($E = 3.0769230 \times 10^{10}$ Pa = 30769.23 MPa).

$A = 7.6^2$ cm^2, $I_z = 7.6^4/12 = 278$ cm^4, $k_z^2 = I_z/A = 2.19^2$, $k_z^3/I_z = 0.0379$.

The intersection of Euler's and RDA curves is adopted on slenderness ratio $\lambda = 100$, what is at the same time the lower boundary of elastic domain on which the proportional stress is determined. Specified intersection is used in calculation of creep coefficient, and further in RDA expression.

$$\varphi = \pi^2 \cdot \frac{k_z^3}{I_z} \cdot \frac{1}{\gamma \cdot \lambda} = \pi^2 \times 0.0379 \times \frac{1}{2.5 \times 10^{-3} \times 100} = 1.5,$$

$$\sigma_{cr}^{RDA} = \frac{E}{\frac{L}{k_z} \cdot \frac{k_z^3}{I_z} \cdot \frac{1}{\gamma \cdot \varphi}} = \frac{30769.23 \times 2.5 \times 10^{-3} \times 1.5}{\frac{L}{k_z} \cdot 0.0379}.$$

The results of this expression are given in the Table 5, for $L = 20.3, 30, \ldots, 450$ cm, and shown in Fig. 13.

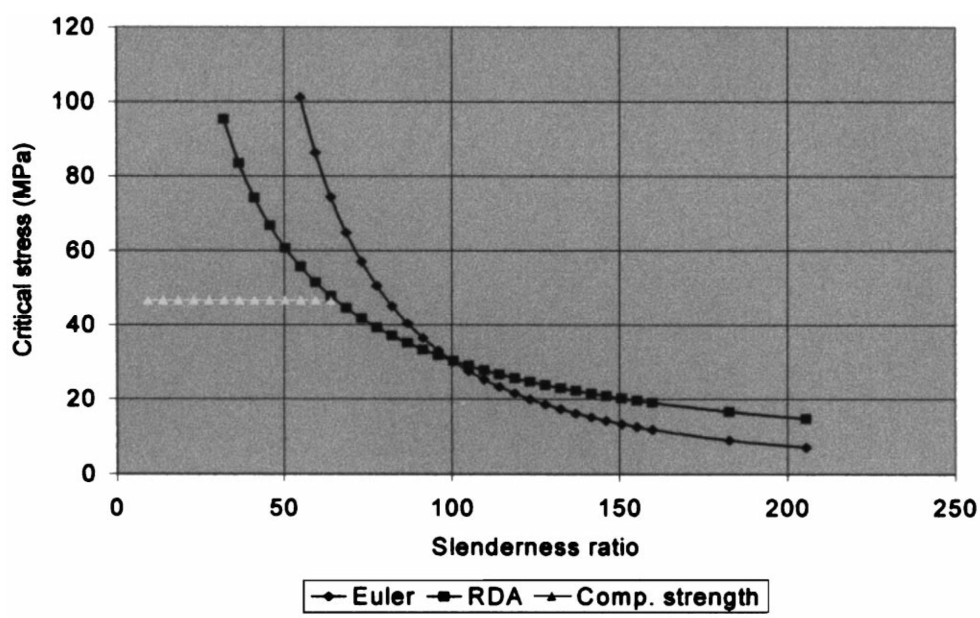


Fig. 13. Prediction of buckling curves of concrete columns by RDA and Euler's formula.

Table 5
Critical stresses of concrete columns

L (cm)	λ	σ_{cr}^{E} (MPa)	σ_{cr}^{RDA} (MPa)	Comp. strength (Hirst and Neville, 1977)
20.3	9.27	3534.37	328.44	46.5
30	13.70	1681.31	222.24	46.5
40	18.26	910.30	166.68	46.5
50	22.83	582.59	133.35	46.5
60	27.40	404.58	111.12	46.5
70	31.96	297.24	95.25	46.5
80	36.53	277.58	83.34	46.5
90	41.10	179.81	74.08	46.5
100	45.66	145.65	66.67	46.5
110	50.23	120.37	60.61	46.5
120	54.79	101.14	55.56	46.5
130	59.36	86.18	51.29	46.5
140	63.93	74.31	47.62	46.5
150	68.49	64.73	44.45	
160	73.06	56.89	41.67	
170	77.63	50.40	39.22	
180	82.19	44.95	37.04	
190	86.76	40.35	35.09	
200	91.32	36.41	33.34	
210	95.89	33.03	31.75	
220	100.46	30.09	30.30	
230	105.02	27.53	28.99	
240	109.59	25.29	27.78	
250	114.16	23.30	26.67	
260	118.72	21.55	25.64	
270	123.29	19.98	24.69	
280	127.85	18.58	23.81	
290	132.42	17.32	22.99	
300	136.99	16.18	22.22	
310	141.55	15.16	21.51	
320	146.12	14.22	20.84	
330	150.68	13.37	20.20	
340	155.25	12.60	19.61	
350	159.82	11.89	19.05	
400	182.65	9.10	16.67	
450	205.48	7.19	14.82	

Because of indirect introduction the creep coefficient in this analysis the discussion about this coefficient in the frame of RDA procedure, is possible only on the base of the experimental investigation.

6. Conclusion

The analogy between the new, rheological model and dynamical model with respect to the viscous damping, it occurs only when there is the arrangement of five fundamental elements making up the body (cf. Eq. (1)). The parameter H' (the slope of the viscoplastic strain), furthermore, must be included in the yield condition. All of the other previously mentioned combinations with less or more fundamental elements cannot restore this analogy. Due to the present observation, fundamentally new, our understanding of rheology as well as mechanics has been achieved.

Generally speaking, the rheological–dynamical analysis is based on new rheological model and has been derived in order to solve the dynamical problems, but can also be used to explain any statical problems, considering the correspondent limit values of presented mathematical expressions.

Consequently, the RDA also improves current understanding of the phenomenon regarding in-plane buckling. This procedure can be used as very successful design method, if the governing parameters of steel, timber or concrete (creep coefficient, specific gravity and modulus of elasticity) are identified correctly. The most important governing parameter — creep coefficient, can be determined on the base of Euler's and RDA curves intersection, the choice of which has to do with the designer's feeling and importance of the project task (here adopted $\lambda = 100$). The stability design of columns is carried out with modified compression strength using rheological–dynamical modulus. In addition to the complexity of the RDA theory and the consequently derived expressions, the final solution (expressions) gives simple design procedure and quite acceptable results, both in theoretical and practical sense.

As a result, the major aim of the paper to check and confirm the RDA as universal procedure for solving the inelastic problems of any kind has been achieved.

Acknowledgements

In summary, the RDA iterative procedure allows us to solve the complete problem of buckling curves of columns when the governing parameters of materials: creep coefficient, specific gravity and modulus of elasticity are identified correctly.

Appendix A. The plots of the logarithm of critical stress versus the logarithm of slenderness ratio

In the plots of the logarithm of critical stress versus the logarithm of slenderness ratio, the Euler's and RDA laws are straight lines. The slope of Euler's lines is $-2/1$. For RDA, the slope of this lines is $-1/1$ (Figs. 14–17).

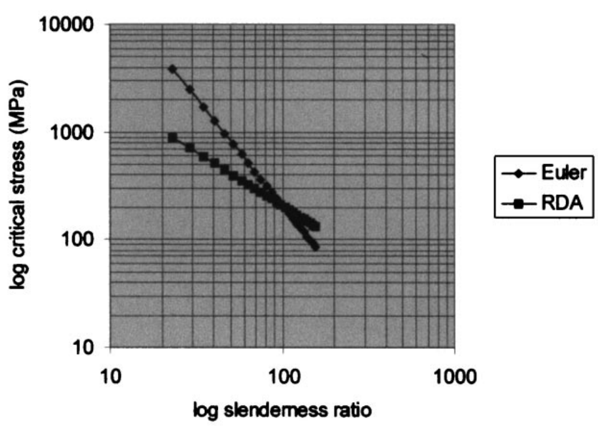


Fig. 14. The plot of the logarithm of critical stress versus the logarithm of slenderness ratio for the Example 1.

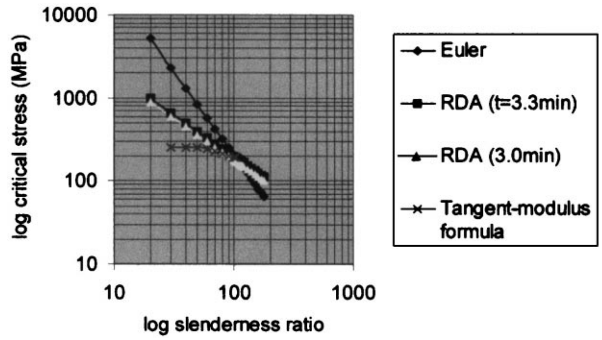


Fig. 15. The plot of the logarithm of critical stress versus the logarithm of slenderness ratio for the Example 2.

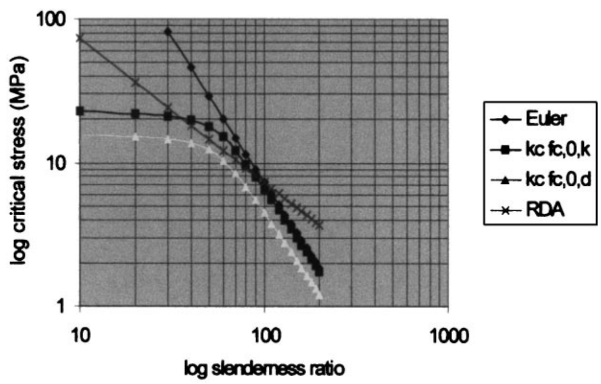


Fig. 16. The plot of the logarithm of critical stress versus the logarithm of slenderness ratio for the Example 3.

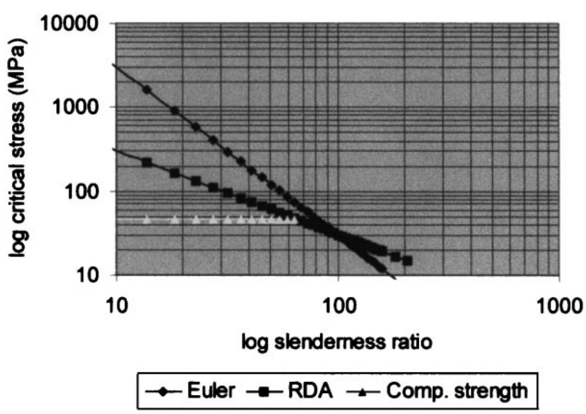


Fig. 17. The plot of the logarithm of critical stress versus the logarithm of slenderness ratio for the Example 4.

Appendix B. Prediction of strength of columns with imperfections in inelastic medium

In order to estimate the buckling load of a compression member showing visible deformations, it is recommended to measure the geometric imperfections. The amplitude of the equivalent sine half-wave can be determined as follows:

$$f_0 = 0.5\left[0.707\left(f_{1/4} + f_{3/4}\right) + f_{1/2}\right], \tag{B1}$$

where $f_{1/2}$, $f_{1/4}$ and $f_{3/4}$ are, respectively, the mid-length and the quarter-point values. The imperfections must be measured in the directions of both principal axes of inertia.

Initial curvature Eq. (B2) is taken for the estimation from above the bending of columns with geometrical imperfections in accordance with the form of column buckling.

$$f = f_0 \sin\frac{n\pi x}{L} \tag{B2}$$

If we assume that an initial curvature of the shorter, stockier column is negligible as compared to its cross-sectional dimensions, we can also assume with sufficient accuracy that after deformation force N will give rise to axial compression only.

Applying the method of superposition of forces, we can find the normal stresses at any point of an arbitrary section as the algebraic sum of stresses caused by forces N and bending of moment $M = Nf$.

Assume Oy and Oz to be the principal axes of inertia of the section. Let f_{0y} and f_{0z} be the coordinates of point A, the point of intersection of the line of geometrical imperfection in the mid-length with the cross-section. In order to find the maximum stressed point of the section, we write the expression for normal stress σ at an arbitrary point B having coordinates y and z. Moment Nf_{0y} bends the column in plane xOy about the neutral axis Oz and gives rise to normal compressive stress $\frac{Nf_{0y}y}{I_z}$ at point B. Similarly, the normal stress at point B due to bending in plane xOz caused by moment Nf_{0z} is also compressive and is equal to $\frac{Nf_{0z}z}{I_y}$.

Summing the stresses due to axial compression and bending in two planes and considering the compressive stresses to be positive, we get the following formula for the stress at point B:

$$\sigma = N\left(\frac{1}{A} + \frac{f_{0y}y}{I_z} + \frac{f_{0z}z}{I_y}\right). \tag{B3}$$

Now we write:

$$k = \frac{1}{\dfrac{L}{E_{\mathrm{H}}A} + \dfrac{Lf_{0y}y}{E_{\mathrm{H}}I_z} + \dfrac{Lf_{0z}z}{E_{\mathrm{H}}I_y}}, \qquad H'^{\mathrm{D}} = \frac{\gamma\varphi(t)}{\dfrac{L}{A} + \dfrac{Lf_{0y}y}{I_z} + \dfrac{Lf_{0z}z}{I_y}} = E_{\mathrm{R}}(t, t_0), \tag{B4}$$

and

$$\sigma_{\mathrm{cr}} = \frac{\gamma\varphi(t)}{\left(\dfrac{L}{k_z}\right)\dfrac{k_z^3}{I_z} + \left(\dfrac{L}{k_z}\right)\dfrac{f_{0y}y}{k_zA} + \left(\dfrac{L}{k_y}\right)\dfrac{f_{0z}z}{k_yA}}E_{\mathrm{H}}(t_0) = \frac{\gamma\varphi(t)}{\left(\dfrac{L}{k_z}\right)\left[\dfrac{k_z^3}{I_z} + \dfrac{1}{A}\left(\dfrac{f_{0y}y}{k_z} + \dfrac{k_zf_{0z}z}{k_y^2}\right)\right]}E_{\mathrm{H}}(t_0). \tag{B5}$$

Appendix C. Improved RDA model for prediction of buckling curves of columns

In Eq. (71), it is suggested that when $\sigma_{cr} \geq \sigma_p$, the modulus $E_H(t)$ is not constant. In fact, $E_H(t)$ is equal to $E_R(t)$ in Fig. 8. As $\sigma(t)$ increases $E_R(t)$ decreases. The use of curves $E_R(t)$ v/s $\sigma(t)$, which are available for the column material is one suggestion to improve the results of the new RDA method. Such a curve can be obtained by using RDA too.

Let us analyze the typical steel column (Example 2) in order to compare the results obtained by improved RDA model and tangent modulus formula (Smith and Sidebottom, 1965).

- Initial elastic modulus, $E_H = 2.1 \times 10^5$ MPa.
- Loading time, $t_p = 3.3$ min ($3.3 \times 60 = 198$ s).
- Creep coefficient, $\varphi(t_p) = 0.470878$.
- Loading stress,

$$\sigma(t) = \sigma_{cr} \cdot \sin(\omega_\sigma \cdot t), \tag{C1}$$

where σ_{cr} are the critical stresses obtained by the new RDA model (Example 2, $t_p = 3.3$ min) for slenderness rations: 30, 40, 50, 60, 70, 80, 90, 100 and 104.

$$\omega_\sigma = \frac{\pi}{2 \cdot t_p} = \frac{\pi}{396}, \tag{C2}$$

- Now, taking into account the viscoelastic deforming we have RDA modulus from Eq. (51),

$$E_R(t) = \frac{\dfrac{1+\delta^2}{E_H} + \dfrac{1}{E_K^D(t)}}{\dfrac{1+\delta^2}{E_H^2} + \dfrac{1}{E_K^D(t)}\left(\dfrac{2}{E_H} + \dfrac{1}{E_K^D(t)}\right)}, \tag{C3}$$

where:

$$\delta = \frac{\omega_\sigma}{\omega} = \omega_\sigma \cdot T_K^D = \frac{\pi}{396} \cdot T_K^D, \tag{C4}$$

$$T_K^D = \sqrt{\frac{m}{k}} = \sqrt{\frac{L \cdot A \cdot \rho}{\frac{E_H \cdot A}{L}}} = L \cdot \sqrt{\frac{\gamma}{E_H \cdot g}} = L \cdot \sqrt{\frac{78.6}{210{,}000{,}000 \times 10}} = L \times 0.000193464, \tag{C5}$$

$$\delta = \frac{\pi}{396} \times L \times 0.000193464 = 0.000001534 \times L,$$

$$\varphi(t) = 2 \times 10^{-9} \times \sigma^2(t) \times \left(\frac{t}{3600}\right)^{1.2} \times 2.1 \times 10^5, \tag{C6}$$

$$E_K^D(t) = \frac{E_H}{\varphi(t)} = \frac{210{,}000}{\varphi(t)}. \tag{C7}$$

Time functions $\sigma(t)$ and $E_R(t)$ are found using Eqs. (C1) and (C3), by writing them in pairs for same times. Such pairs are listed in Table 6 and Fig. 18 for slenderness ratios 104 and 30.

Table 6
Time functions: $\sigma(t)$ and $E_R(t)$

t	$\lambda = 104$				t	$\lambda = 30$			
	$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$		$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$
1	1.520245	5.2422E−08	4.01E+12	210000	1	5.270046	6.29963E−07	3.33353E+11	209999.9
2	3.040395	4.8171E−07	4.36E+11	209999.9	2	10.53976	5.78874E−06	36277352547	209998.8
3	4.560353	1.7629E−06	1.19E+11	209999.6	3	15.80881	2.11851E−05	9912634965	209995.6
4	6.080024	4.4256E−06	4.75E+10	209999.1	4	21.07687	5.31827E−05	3948652702	209988.8
5	7.599312	9.0365E−06	2.32E+10	209998.1	5	26.3436	0.000108593	1933832716	209977.2
6	9.118123	1.6191E−05	1.3E+10	209996.6	6	31.60867	0.000194572	1079292480	209959.1
7	10.63636	2.6509E−05	7.92E+09	209994.4	7	36.87175	0.00031856	659216228.5	209933.1
8	12.15393	4.0628E−05	5.17E+09	209991.5	8	42.13251	0.000488235	430120715.3	209897.5
9	13.67073	5.9205E−05	3.55E+09	209987.6	9	47.39062	0.000711479	295159811.2	209850.7
10	15.18667	8.2911E−05	2.53E+09	209982.6	10	52.64575	0.000996352	210768852.7	209791
11	16.70165	0.00011243	1.87E+09	209976.4	11	57.89756	0.001351071	155432295.7	209716.7
12	18.21559	0.00014845	1.41E+09	209968.8	12	63.14573	0.001783987	117713843.3	209626
13	19.72838	0.00019169	1.1E+09	209959.8	13	68.38993	0.002303576	91162605.13	209517.4
14	21.23992	0.00024285	8.65E+08	209949	14	73.62982	0.002918417	71956814.62	209388.9
15	22.75013	0.00030267	6.94E+08	209936.5	15	78.86507	0.003637184	57736981.02	209239
16	24.25891	0.00037185	5.65E+08	209921.9	16	84.09537	0.00446863	46994267.71	209065.8
17	25.76616	0.00045115	4.65E+08	209905.3	17	89.32037	0.005421582	38734084.39	208867.6
18	27.27179	0.0005413	3.88E+08	209886.4	18	94.53975	0.006504923	32283240.91	208642.8
19	28.77571	0.00064305	3.27E+08	209865	19	99.75318	0.00772759	27175354.48	208389.7
20	30.27781	0.00075713	2.77E+08	209841.1	20	104.9603	0.009098559	23080576.78	208106.5
21	31.778	0.00088431	2.37E+08	209814.5	21	110.1609	0.010626841	19761282.09	207791.8
22	33.2762	0.00102532	2.05E+08	209784.9	22	115.3545	0.012321469	17043423.02	207444
23	34.7723	0.00118094	1.78E+08	209752.3	23	120.5408	0.014191493	14797597.21	207061.5
24	36.26622	0.0013519	1.55E+08	209716.5	24	125.7196	0.016245975	12926278.92	206642.9
25	37.75785	0.00153897	1.36E+08	209677.3	25	130.8905	0.018493974	11355049.79	206186.8
26	39.2471	0.00174289	1.2E+08	209634.6	26	136.0531	0.020944548	10026475.53	205691.9
27	40.73389	0.00196442	1.07E+08	209588.3	27	141.2071	0.02360674	8895764.681	205156.9
28	42.21811	0.00220431	95267737	209538.1	28	146.3523	0.026489573	7927647.684	204580.7
29	43.69967	0.00246332	85250919	209484	29	151.4882	0.029602046	7094104.181	203962.3
30	45.17848	0.00274217	76581554	209425.7	30	156.6147	0.032953126	6372688.212	203300.6
31	46.65445	0.00304163	69041900	209363.2	31	161.7312	0.036551741	15745280.367	202594.8
32	48.12749	0.00336243	62454916	209296.3	32	166.8376	0.040406773	5197148.457	201844.1
33	49.59749	0.00370529	56675690	209224.8	33	171.9335	0.044527056	4716233.637	201047.9
34	51.06438	0.00407096	51584857	209148.6	34	177.0186	0.048921365	4292603.003	200205.7
35	52.52805	0.00446016	47083513	209067.5	35	182.0925	0.053598415	3918026.317	199316.9
36	53.98841	0.00487361	43089250	208981.5	36	187.1549	0.05856685	3585646.126	198381.4
37	55.44538	0.00531201	39533046	208890.4	37	192.2056	0.063835244	3289718.771	197399
38	56.89885	0.00577609	36356803	208794	38	197.2442	0.069412089	3025409.614	196369.6
39	58.34875	0.00626653	33511388	208692.2	39	202.2704	0.075305793	2788630.094	195293.3
40	59.79497	0.00678403	30955065	208585	40	207.2838	0.081524675	2575907.223	194170.3
41	61.23743	0.00732927	28652234	208472.1	41	212.2842	0.08807696	2384278.485	193001.1
42	62.67603	0.00790294	26572404	208353.4	42	217.2712	0.094970769	2211206.685	191785.9
43	64.11069	0.00850569	24689364	208228.9	43	222.2446	0.102214122	2054510.622	190525.6
44	65.54132	0.00913818	22980497	208098.4	44	227.204	0.109814927	1912308.341	189220.7
45	66.96782	0.00980107	21426225	207961.8	45	232.1491	0.117780975	1782970.47	187872.2
46	68.39011	0.010495	20009537	207818.9	46	237.0795	0.12611994	1665081.669	186481
47	69.80809	0.01122058	18715614	207669.8	47	241.9951	0.13483937	1557408.643	185048.2
48	71.22168	0.01197844	17531502	207514.3	48	246.8954	0.143946684	1458873.486	183575
49	72.63078	0.01276918	16445848	207352.3	49	251.7801	0.153449166	1368531.39	182062.6

Table 6 (*continued*)

t	$\lambda = 104$				t	$\lambda = 30$			
	$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$		$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$
50	74.03532	0.0135934	15448671	207183.7	50	256.6491	0.163353963	1285551.917	180512.6
51	75.43519	0.01445169	14531177	207008.4	51	261.5018	0.173668078	1209203.223	178926.2
52	76.83032	0.0153446	13685596	206826.3	52	266.3382	0.184398369	1138838.708	177305.2
53	78.22061	0.0162727	12905046	206637.4	53	271.1577	0.195551539	1073885.694	175651.1
54	79.60598	0.01723654	12183417	206441.7	54	275.9602	0.207134138	1013835.777	173965.8
55	80.98634	0.01823665	11515274	206238.9	55	280.7453	0.219152556	958236.5987	172250.8
56	82.3616	0.01927354	10895768	206029.1	56	285.5128	0.231613019	906684.785	170508.1
57	83.73168	0.02034772	10320568	205812.2	57	290.2622	0.244521584	858819.8916	168739.5
58	85.09648	0.02145967	9785796	205588.1	58	294.9935	0.257884138	814319.1811	166947
59	86.45594	0.02260988	9287973	205356.9	59	299.7061	0.271706392	772893.1148	165132.5
60	87.80995	0.02379881	8823971	205118.4	60	304.3999	0.28599388	734281.4479	163297.8
61	89.15843	0.02502689	8390974	204872.7	61	309.0745	0.300751949	698249.8397	161445.1
62	90.50131	0.02629457	7986441	204619.6	62	313.7297	0.315985763	664586.9035	159576.2
63	91.83848	0.02760224	7608078	204359.2	63	318.3651	0.331700297	633101.634	157693.1
64	93.16988	0.02895032	7253806	204091.5	64	322.9805	0.34790033	603621.1571	155797.9
65	94.49541	0.03033918	6921744	203816.4	65	327.5756	0.364590448	575988.7597	153892.3
66	95.815	0.03176918	6610180	203533.9	66	332.15	0.381775035	550062.1589	151978.4
67	97.12856	0.03324068	6317560	203244	67	336.7035	0.399458274	525711.9802	150058.1
68	98.436	0.03475401	6042469	202946.8	68	341.2359	0.417644141	502820.4148	148133.1
69	99.73725	0.03630947	5783615	202642.2	69	345.7468	0.436336404	481280.035	146205.3
70	101.0322	0.03790737	5539819	202330.2	70	350.2359	0.455538622	460992.7454	144276.5
71	102.3208	0.03954799	5310005	202010.9	71	354.7029	0.475254136	441868.8528	142348.4
72	103.603	0.04123158	5093184	201684.2	72	359.1477	0.495486073	423826.2411	140422.6
73	104.8787	0.04295838	4888452	201350.3	73	363.5698	0.51623734	406789.6367	138500.7
74	106.1477	0.04472862	4694980	201009.1	74	367.9691	0.537510622	390689.954	136584.4
75	107.4101	0.04654251	4512004	200660.7	75	372.3452	0.559308379	375463.7121	134675.1
76	108.6657	0.04840023	4338822	200305.2	76	376.6979	0.581632845	361052.5121	132774.2
77	109.9145	0.05030194	4174789	199942.5	77	381.0268	0.604486028	347402.5704	130883
78	111.1563	0.0522478	4019308	199572.8	78	385.3318	0.627869702	334464.2995	129003
79	112.3912	0.05423793	3871829	199196	79	389.6125	0.651785408	322191.932	127135.2
80	113.6189	0.05627245	3731844	198812.3	80	393.8687	0.676234456	310543.1825	125280.8
81	114.8396	0.05835143	3598884	198421.8	81	398.1002	0.701217915	299478.9431	123441
82	116.053	0.06047495	3472512	198024.5	82	402.3065	0.726736619	288963.0088	121616.7
83	117.2591	0.06264307	3352326	197620.4	83	406.4876	0.752791161	278961.8302	119808.9
84	118.4578	0.0648558	3237953	197209.8	84	410.643	0.779381893	269444.2891	118018.5
85	119.6491	0.06711316	3129044	196792.6	85	414.7726	0.806508924	260381.4959	116246.3
86	120.8328	0.06941513	3025277	196369	86	418.8761	0.834172119	251746.6064	114493.1
87	122.0089	0.07176169	2926352	195939.1	87	422.9533	0.862371097	243514.6547	112759.5
88	123.1774	0.07415279	2831991	195502.9	88	427.0038	0.891105234	235662.4022	111046.2
89	124.3381	0.07658834	2741932	195060.6	89	431.0275	0.920373654	228168.2002	109353.7
90	125.491	0.07906826	2655933	194612.3	90	435.024	0.950175234	221011.8643	107682.6
91	126.6359	0.08159244	2573768	194158.2	91	438.9931	0.980508605	214174.5609	106033.4
92	127.7729	0.08416073	2495226	193698.2	92	442.9346	1.011372144	207638.7027	104406.3
93	128.9019	0.08677298	2420108	193232.6	93	446.8483	1.04276398	201387.8538	102801.9
94	130.0228	0.08942902	2348231	192761.5	94	450.7338	1.07468199	195406.643	101220.3
95	131.1354	0.09212864	2279421	192285	95	454.591	1.1071238	189680.6843	99661.92
96	132.2398	0.09487164	2213517	191803.3	96	458.4195	1.140086784	184196.5041	98126.86
97	133.3359	0.09765776	2150367	191316.5	97	462.2192	1.173568065	178941.4745	96615.33
98	134.4236	0.10048676	2089828	190824.6	98	465.9898	1.207564512	173903.7524	95127.46
99	135.5029	0.10335834	2031766	190328	99	469.731	1.242072745	169072.2229	93663.33

(continued on next page)

Table 6 (*continued*)

t	$\lambda = 104$				t	$\lambda = 30$			
	$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$		$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$
100	136.5736	0.10627221	1976058	189826.7	100	473.4427	1.27708913	164436.4477	92223
101	137.6357	0.10922804	1922583	189320.9	101	477.1246	1.312609781	159986.6182	90806.5
102	138.6892	0.11222549	1871233	188810.6	102	480.7765	1.348630563	155713.511	89413.81
103	139.7339	0.11526419	1821902	188296.2	103	484.3981	1.385147087	151608.4479	88044.88
104	140.7698	0.11834375	1774492	187777.7	104	487.9893	1.422154716	147663.2589	86699.66
105	141.7969	0.12146378	1728910	187255.3	105	491.5497	1.459648562	143870.2476	85378.05
106	142.815	0.12462384	1685071	186729.1	106	495.0792	1.497623489	140222.1597	84079.93
107	143.8242	0.12782349	1642891	186199.3	107	498.5775	1.536074111	136712.1537	82805.15
108	144.8243	0.13106225	1602292	185666.2	108	502.0444	1.574994797	133333.7738	81553.56
109	145.8153	0.13433964	1563202	185129.7	109	505.4798	1.614379668	130080.9247	80324.98
110	146.7971	0.13765514	1525551	184590.2	110	508.8833	1.654222599	126947.8486	79119.21
111	147.7697	0.14100824	1489275	184047.8	111	512.2548	1.694517225	123929.1032	77936.04
112	148.7329	0.14439837	1454310	183502.5	112	515.5941	1.735256934	121019.5424	76775.24
113	149.6869	0.14782497	1420599	182954.7	113	518.9009	1.776434875	118214.297	75636.57
114	150.6314	0.15128744	1388086	182404.5	114	522.1751	1.818043958	115508.7582	74519.77
115	151.5664	0.15478518	1356719	181852	115	525.4164	1.860076854	112898.5609	73424.6
116	152.4918	0.15831756	1326448	181297.4	116	528.6246	1.902525999	110379.5691	72350.77
117	153.4077	0.16188393	1297226	180740.9	117	531.7995	1.945383592	107947.8622	71298.01
118	154.3139	0.16548362	1269008	180182.7	118	534.941	1.988641603	105599.7218	70266.04
119	155.2104	0.16911594	1241752	179622.9	119	538.0489	2.032291771	03331.6196	69254.55
120	156.0972	0.17278019	1215417	179061.7	120	541.1228	2.076325602	101140.2064	68263.26
121	156.9741	0.17647564	1189966	178499.2	121	544.1627	2.120734383	99022.30174	67291.85
122	157.8411	0.18020155	1165362	177935.7	122	547.1684	2.165509174	96974.88355	66340.04
123	158.6982	0.18395715	1141570	177371.3	123	550.1396	2.210640814	94995.07955	65407.5
124	159.5454	0.18774167	1118558	176806.1	124	553.0762	2.256119923	93080.15848	64493.94
125	160.3824	0.1915543	1096295	176240.4	125	555.9779	2.301936906	91227.52212	63599.03
126	161.2094	0.19539423	1074750	175674.3	126	558.8447	2.348081953	89434.69786	62722.48
127	162.0262	0.19926063	1053896	175107.9	127	561.6763	2.394545045	87699.33164	61863.96
128	162.8329	0.20315265	1033705	174541.4	128	564.4726	2.441315954	86019.18143	61023.17
129	163.6293	0.20706941	1014153	173975.1	129	567.2333	2.48838425	84392.11107	60199.79
130	164.4154	0.21101003	995213.4	173409	130	569.9584	2.535739298	82816.08452	59393.52
131	165.1911	0.21497361	976864.1	172843.3	131	572.6475	2.583370266	81289.16042	58604.05
132	165.9564	0.21895924	959082.6	172278.1	132	575.3007	2.631266128	79809.48705	57831.07
133	166.7113	0.22296597	941847.8	171713.7	133	577.9176	2.679415665	78375.29754	57074.28
134	167.4558	0.22699287	925139.2	171150.1	134	580.4981	2.72780747	76984.90539	56333.38
135	168.1896	0.23103896	908937.6	170587.6	135	583.0422	2.776429951	75636.70027	55608.08
136	168.9129	0.23510326	893224.5	170026.3	136	585.5495	2.825271334	74329.14407	54898.07
137	169.6256	0.23918479	877982.2	169466.3	137	588.0199	2.874319669	73060.76714	54203.06
138	170.3275	0.24328253	863193.9	168907.7	138	590.4534	2.923562831	71830.16482	53522.78
139	171.0188	0.24739546	848843.4	168350.8	139	592.8497	2.972988526	70635.99411	52856.94
140	171.6993	0.25152254	834915.2	167795.6	140	595.2087	3.022584293	69476.97057	52205.25
141	172.369	0.25566273	821394.7	167242.4	141	597.5302	3.072337511	68351.8654	51567.44
142	173.0278	0.25981495	808267.6	166691.1	142	599.8141	3.122235397	67259.50266	50943.23
143	173.6758	0.26397814	795520.4	166142.1	143	602.0603	3.172265026	6198.75663	50332.37
144	174.3128	0.26815119	783140.3	165595.4	144	604.2685	3.222413293	65168.54943	49734.59
145	174.9388	0.27233303	771114.7	165051.1	145	606.4388	3.27266699	64167.84861	49149.63
146	175.5539	0.27652252	759431.8	164509.4	146	608.5708	3.323012746	3195.66502	48577.23
147	176.1579	0.28071854	748080.3	163970.5	147	610.6646	3.373437038	62251.05068	48017.15
148	176.7508	0.28491997	737049.1	163434.3	148	612.72	3.423926244	61333.09687	47469.15
149	177.3325	0.28912566	726327.8	162901.1	149	614.7367	3.474466595	60440.93223	46932.97

Table 6 (*continued*)

t	$\lambda = 104$				t	$\lambda = 30$			
	$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$		$\sigma(t)$	$\varphi(t)$	$E_K^D(t)$	$E_R(t)$
150	177.9031	0.29333444	715906.4	162371	150	616.7148	3.525044202	59573.72105	46408.39
151	178.4626	0.29754516	705775.2	161844.1	151	618.6541	3.575645065	8730.66159	45895.17
152	179.0108	0.30175664	695925	161320.5	152	620.5544	3.62625505	57910.9845	45393.09
153	179.5477	0.3059677	686347	160800.3	153	622.4157	3.676859945	57113.95135	44901.92
154	180.0733	0.31017714	677032.5	160283.7	154	624.2378	3.727445414	56338.85324	4421.45
155	180.5876	0.31438376	667973.4	159770.7	155	626.0206	3.777997035	5585.00929	43951.47
156	181.0905	0.31858636	659161.9	159261.5	156	627.764	3.828500269	54851.76577	43491.77
157	181.582	0.32278371	650590.4	158756.1	157	629.468	3.878940521	54138.49448	43042.13
158	182.0621	0.32697461	642251.7	158254.7	158	631.1322	3.929303093	53444.59183	42602.37
159	182.5308	0.3311578	634138.8	157757.4	159	632.7568	3.979573213	52769.47772	42172.29
160	182.9879	0.33533207	626244.9	157264.3	160	634.3416	4.029736036	52112.5945	41751.69
161	183.4335	0.33949617	618563.7	156775.4	161	635.8864	4.079776652	51473.40601	41340.4
162	183.8676	0.34364886	611088.9	156290.8	162	637.3912	4.129680084	50851.39665	40938.23
163	184.2902	0.34778887	603814.6	155810.8	163	638.8559	4.179431304	50246.07051	40544.99
164	184.7011	0.35191497	596735.1	155335.2	164	640.2803	4.229015228	49656.95054	40160.53
165	185.1004	0.35602589	589844.7	154864.3	165	641.6645	48525.51036	49083.57772	39784.66
166	185.488	0.36012036	583138.4	154398.1	166	643.0083	4.327620636	48525.51036	39417.22
167	185.864	0.36419713	576610.8	153936.7	167	644.3116	4.376611748	47982.32333	39058.06
168	186.2283	0.36825492	570257.1	153480.2	168	645.5744	4.425374831	47453.60744	38707
169	186.5808	0.37229246	564072.7	153028.6	169	646.7966	4.473894629	46938.96871	38363.91
170	186.9216	0.37630849	558052.8	152582.1	170	647.978	4.522155866	46438.02784	38028.63
171	187.2507	0.38030174	552193.1	152140.6	171	649.1187	4.570143256	45950.41955	37701.01
172	187.5679	0.38427092	546489.4	151704.4	172	650.2185	4.617841504	45475.79206	37380.91
173	187.8734	0.38821476	540937.7	151273.4	173	651.2773	4.665235316	45013.80655	37068.19
174	188.167	0.39213201	535533.9	150847.8	174	652.2952	4.7123094	44564.13664	36762.71
175	188.4488	0.39602137	530274.4	150427.5	175	653.2721	4.759048478	44126.46792	36464.36
176	188.7187	0.39988159	525155.5	150012.7	176	654.2078	4.805437285	43700.49749	36172.99
177	188.9768	0.40371139	520173.6	149603.4	177	655.1023	4.851460579	43285.9335	35888.48
178	189.2229	0.40750951	515325.4	149199.7	178	655.9557	4.897103146	42882.49476	35610.7
179	189.4572	0.41127468	510607.7	148801.6	179	656.7677	4.942349807	42489.9103	35339.56
180	189.6795	0.41500565	506017.2	148409.3	180	657.5384	4.987185421	42107.91905	35074.91
181	189.8899	0.41870116	501551	148022.7	181	658.2677	5.031594891	41736.26942	34816.66
182	190.0883	0.42235995	497206.2	147642	182	658.9556	5.075563175	41374.71898	34564.7
183	190.2748	0.42598079	492980	147267.1	183	659.602	5.1190752844	1023.03411	34318.91
184	190.4493	0.42956242	488869.6	146898.1	184	660.2069	5.162116295	40680.98973	34079.2
185	190.6118	0.43310361	484872.4	146535.1	185	660.7702	5.204671353	40348.36895	33845.47
186	190.7623	0.43660314	480986	146178.2	186	661.292	5.246725676	40024.9628	33617.61
187	190.9008	0.44005977	477207.9	145827.3	187	661.7721	5.288264565	39710.56996	33395.54
188	191.0273	0.4434723	473535.8	145482.5	188	662.2106	5.329273406	39404.99651	33179.16
189	191.1417	0.44683951	469967.4	145143.9	189	662.6074	5.369737678	39108.05566	32968.39
190	191.2442	0.4501602	466500.6	144811.6	190	662.9625	5.409642958	38819.56751	32763.14
191	191.3346	0.45343319	463133.3	144485.5	191	663.2759	5.448974925	38539.35885	32563.31
192	191.4129	0.45665728	459863.5	144165.7	192	663.5476	5.487719372	38267.26291	32368.85
193	191.4793	0.45983132	456689.2	143852.2	193	663.7775	5.525862203	38003.1192	32179.66
194	191.5335	0.46295412	453608.7	143545.2	194	663.9656	5.563389448	37746.77325	31995.66
195	191.5757	0.46602455	450620	143244.5	195	664.1119	5.60028726	37498.07649	31816.8
196	191.6059	0.46904146	447721.6	142950.4	196	664.2164	5.636541927	37256.886	31642.99
197	191.624	0.47200373	444911.7	142662.7	197	664.2791	5.672139876	37023.06442	31474.16
198	191.63	0.47491022	442188.8	142381.5	198	664.3	5.707067677	36796.47971	31310.26

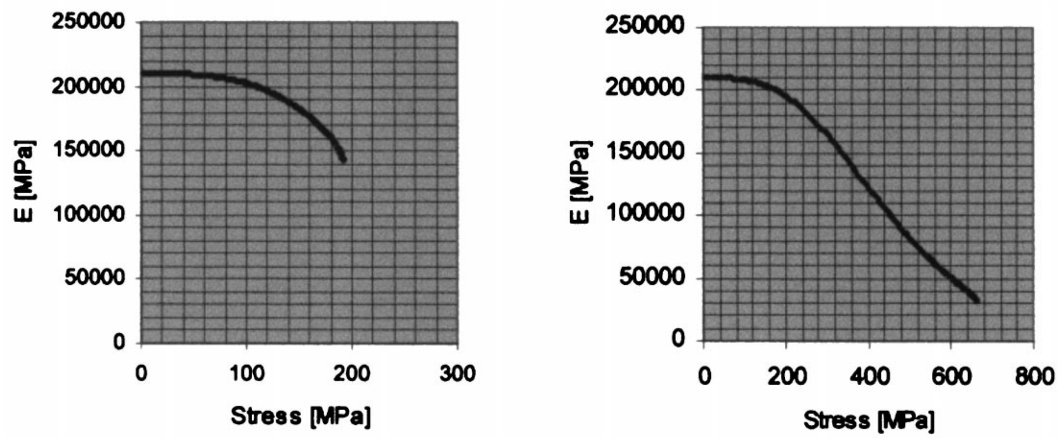


Fig. 18. The plots of the modulus $E_R(t)$ versus the stress $\sigma(t)$: (a) slenderness ratio $\lambda = 104$, (b) slenderness ratio $\lambda = 30$.

(a) slenderness ratio $\lambda = 104$: $L = 1080.46$ cm, $\sigma(t) = 191.63\,\sin(\pi t/396)$, $T_K^D = 0.000193464 \times L = 0.000193464 \times 10.8046 = 0.002090301$ s, $\delta = \omega_\sigma T_K^D = (\pi/396) \times 0.002090301 = 0.000016583$.
(b) slenderness ratio $\lambda = 30$: $L = 311.67$ cm, $\sigma(t) = 664.30\,\sin(\pi t/396)$, $T_K^D = 0.000193464 \times L = 0.000193464 \times 3.1167 = 0.000602969$ s, $\delta = \omega_\sigma T_K^D = (\pi/396) \times 0.000602969 = 0.000004783$.

Fig. 18 shows $E_R(t)$ vs. $\sigma(t)$ curves in certain time interval, $t_p = 3.3$ min ($3.3 \times 60 = 198$ s), but different fixed values of critical stresses σ_{cr}. The reduction of the modulus of elasticity is more pronounced in case of shorter columns.

For a computed value of σ_{cr} we can select the appropriate E_R and then recalculate the new σ_{cr}. This iterative procedure must be used until there is convergence.

(a) slenderness ratio $\lambda = 104$
$\sigma_{cr} = 191.63$ MPa
From the Table 6 there is appropriate $E_R = 142381.50$ MPa.
$\sigma_{cr}^{RDA(1)} = 142381.50 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 130.00$ MPa.
$E_R = 192761.50$ MPa.
$\sigma_{cr}^{RDA(2)} = 192761.50 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 176.00$ MPa.
$E_R = 163970.50$ MPa.
$\sigma_{cr}^{RDA(3)} = 163970.50 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 150.00$ MPa.
$E_R = 182954.70$ MPa.
$\sigma_{cr}^{RDA(4)} = 182954.70 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 167.00$ MPa.
$E_R = 171150.00$ MPa.
$\sigma_{cr}^{RDA(5)} = 171150.00 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 156.00$ MPa.
$E_R = 179061.70$ MPa.
$\sigma_{cr}^{RDA(6)} = 179061.70 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 163.00$ MPa.
$E_R = 173975.10$ MPa.
$\sigma_{cr}^{RDA(7)} = 173975.10 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 159.00$ MPa.
$E_R = 177371.30$ MPa.
$\sigma_{cr}^{RDA(8)} = 177371.30 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 161.85$ MPa.
$E_R = 175108.00$ MPa.

$\sigma_{cr}^{RDA(9)} = 175108.00 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 159.80$ MPa.
$E_R = 176806.10$ MPa.
$\sigma_{cr}^{RDA(10)} = 176806.10 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 161.30$ MPa.
$E_R = 175674.00$ MPa.
$\sigma_{cr}^{RDA(11)} = 175674.00 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 160.30$ MPa.
$E_R = 175674.00$ MPa.
$\sigma_{cr}^{RDA(12)} = 176240.40 \times 7.86 \times 10^{-3} \times 0.470878/(104 \times 0.039) = 160.80$ MPa.

(b) slenderness ratio $\lambda = 30$
$\sigma_{cr} = 664.30$ MPa.
From the Table 6 there is appropriate $E_R = 31310.26$ MPa.
$\sigma_{cr}^{RDA(1)} = 31310.26 \times 7.86 \times 10^{-3} \times 0.470878/(30 \times 0.039) = 99$ MPa.
$E_R = 208389.70$ MPa.
$\sigma_{cr}^{RDA(2)} = 208389.70 \times 7.86 \times 10^{-3} \times 0.470878/(30 \times 0.039) = 659$ MPa.
$E_R = 34318.90$ MPa.
$\sigma_{cr}^{RDA(3)} = 34318.90 \times 7.86 \times 10^{-3} \times 0.470878/(30 \times 0.039) = 108.56$ MPa.
$E_R = 207791.80$ MPa.
$\sigma_{cr}^{RDA(4)} = 207791.80 \times 7.86 \times 10^{-3} \times 0.470878/(30 \times 0.039) = 657.31$ MPa.
$E_R = 35074.90$ MPa.

Table 7
Critical stresses

λ	σ_{cr}^{E} (MPa)	σ_{cr}^{RDA} 3.3 min (MPa)	σ_{cr}^{RDA} 3.0 min (MPa)	Tangent modulus formula (MPa)	σ_{cr}^{RDA} 3.3 min iterative procedure (MPa)	σ_{cr}^{RDA} 3.3 min iterative procedure (MPa)	σ_{cr}^{RDA} 3.3 min iterative procedure with constant values after yield stress (MPa)
30		664.3	596.77	256.15	657	111	248
40		498.23	447.58	256	480	149	248
50		398.58	358.06	252	360	188	248
60	575.73	332.15	298.3849	244.77	248	240	248
70	422.98	284.7001	255.7585	228	218	218	218
80	323.8464	249.1126	223.7887	212	197	197	197
90	255.8786	221.4334	198.9233	201.5	180	180	180
100	207.2617	199.2901	179.0309	193	165	165	165
102	199.2135	195.3824	175.5205	192	163	163	163
104	191.6251	191.6251	172.1451	190.99	161	161	161
106	184.4622	188.0095	168.8971				
108	177.6935	184.5278	165.7694				
110	171.2907	181.1728	162.7554				
112	165.2278	177.9376	159.849				
114	159.4811	174.8158	157.0447				
116	154.0292	171.8018	154.337				
118	148.8521	168.8899	151.7211				
120	143.9317	166.075	149.1924				
130	122.6401	153.3	137.7161				
140	105.7458	142.35	127.8792				
150	92.11631	132.86	119.354				
160	80.9616	124.5563	111.8943				
170	71.71685	117.2294	105.3123				
180	63,96966	110,7167	99,46163				

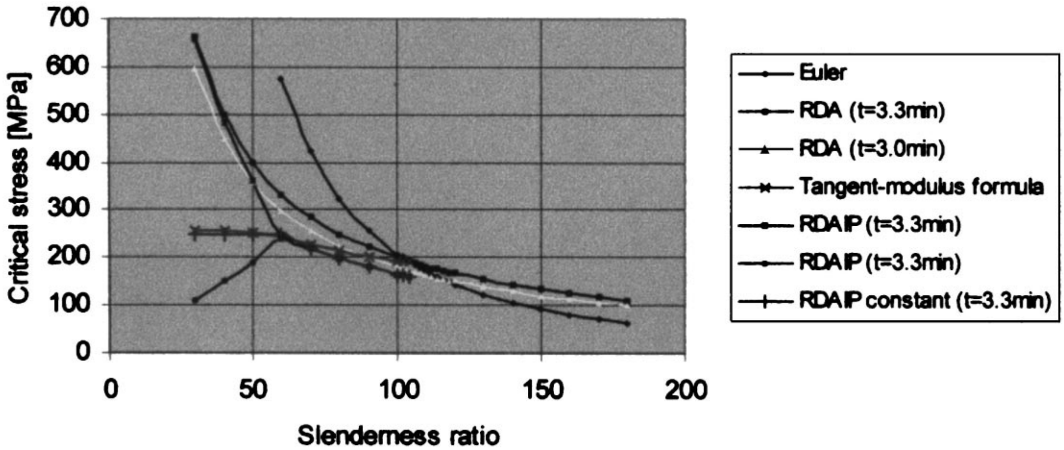


Fig. 19. Comparisons of the RDAIP (iterative procedure), with data of Smith and Sidebottom (1965) (tangent-modulus formula).

$\sigma_{cr}^{RDA(5)} = 35074.90 \times 7.86 \times 10^{-3} \times 0.470878/(30 \times 0.039) = 110.95$ MPa.
$E_R = 207791.80$ MPa.

The curves in Fig. 19 show the variation of critical stresses as the slenderness ratio is raised from 30 to 180 (see Table 7).

From the previous calculations, it is clear that the differences between RDA iterative procedure with constant values after yield stress and tangent-modulus formula are negligible.

In summary, the RDA iterative procedure allows us to solve the complete problem of buckling curves of columns when the governing parameters of materials: creep coefficient, specific gravity and modulus of elasticity are identified correctly.

References

Beedle, L.S., 1960. Basic column strength. Proceedings of the ASCE, Proc. Paper 2555, vol. 86, ST 7.

Blass, H.J., 1995. Columns, STEP 1 (Structural Timber Education Program), Lecture B6, Centrum Hout, The Netherlands, pp. B6/1–8.

Blass, H.J., 1991. Design of columns. In: Proceedings of the International Timber Engineering Conference, London, UK, 1.75–1.81.

Blass, H.J., 1986. Strength model for glulam columns. In: Proceedings of the CIB-W18, Paper 19-12-2, Florence.

Blass, H.J., 1987. Design of timber columns. In: Proceedings of the CIB-W18, Paper 20-2-2, Dublin.

Hirst, G.A., Neville, A.M., 1977. Activation energy of creep of concrete under short-term static and cyclic stresses. Mag. Concr. Res 29 (98), 13–18.

Kojić, M., 1997. Computational procedures in inelastic analysis of solids and structures. Center for Scientific Research of Serbian Academy of Sciences and Arts and University of Kragujevac, YU.

Milašinović, D.D., 1996. Rheological–dynamical analogy. Scientific Meetings of the Serbian Academy of Sciences and Arts, Department of Technical Sciences, Reprint (in Serbian), vol. LXXXIII, Book 2, Belgrade, YU, pp. 103–110.

Milašinović, D.D., 1997. The Finite Strip Method in Computational Mechanics, Faculties of Civil Engineering: Subotica, Budapest, Belgrade, 416. p.

Reiner, M., 1955. Rhe'ologie the'orique, Dunod, Paris, France.

Smith, J.O., Sidebottom, O.M., 1965. Inelastic Behavior of Load-Carrying Members. Wiley, New York, London, Sydney.

Timosheko, S., 1958. Strength of Materials, Part I: Elementary Theory and Problems. Van Nostrand, New York (translated from English into Serbo-Croatian, GK Belgrade, 1965).